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KINEMATICS OF MACHINES

AN ELEMENTARY TEXT-BOOK

BY

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PREFACE.

THIS book is intended as a brief manual for Engineering Students, and treats chiefly of those portions of the subject of the Kinematics of Machines which are likely to be of assistance in the study of the Dynamics of Machines and in work in Machine Design.

The author wishes to thank his friends and colleagues, Dr. E. G. Coker and Mr. H. M. Jaquays, for their kindly criticism and for their help in revision of the proof-sheets.

Many earlier works have been consulted in the preparation of this volume ; wherever possible they are named in the text or in foot-notes. Professors John H. Barr and C. W. MacCord have courteously permitted the use of certain of their diagrams, and the author is indebted to The American Stoker Company, The Brown and Sharpe Manufacturing Company, and The Link Belt Engineering Company for the use of figures and information.

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KINEMATICS OF MACHINES.

CHAPTER I.

INTRODUCTORY CONSIDERATIONS.

1. **Study of Machines.** — In general the study of a Machine involves problems of three distinct kinds. We may first of all consider from a geometrical point of view the motion of any part of the machine with reference to any other part, without taking account of any of the forces acting on such parts. Or, the action of the forces impressed on the parts of the machine, and of the forces due to its own inertia or to the weight of its parts, may be dealt with, and the resulting transformations of energy may be determined. A third branch of the theory of machines treats of the action of these loads and forces in producing stresses and strains in the materials employed in the construction of the machine, and discusses the sizes, forms, and proportions of the various parts which are required either to insure proper strength while avoiding waste of material, or to make the machine capable of doing the work for which it is being designed.

The science dealing with the first-named class of problem is termed the *Kinematics of Machines*, which we may define as being that science which treats of the relative motion of the parts of machines, without regard to the forces producing such motions, or to the stresses and strains produced by such forces.

With this limitation, in the case of almost all bodies forming portions of machines, it is possible to neglect any deformation they may undergo in working, and in studying the Kinematics of Machines we may at once apply to machine problems the results obtained by the study of the motion of rigid bodies. Important exceptions will present themselves to the reader's mind; for example, ropes, belts, and springs cannot be considered kinematically as being rigid, and many mechanical contrivances involve the use of liquid or gaseous material. Such cases as these will be considered later.

By the term *Machine* we may understand a combination or arrangement of certain portions of resistant material, the relative motions of which are controlled in such a way that some form of available energy is transmitted from place to place, or is transformed into another desired kind. This definition includes under the head of Machines all contrivances which have for their object the transformation or transmission of energy, or the performance of some particular kind of work, and further implies that a single portion of material is not considered as a machine. The so-called *simple machines* in every case involve the idea of more than one piece of material.

A combination or arrangement of portions of material by means of which forces are transmitted or loads are carried without sensible relative motions of the component parts is called a *Structure*.

The term *Mechanism* is often used as an equivalent for the word Machine. It is, however, preferable to restrict its use somewhat, and to employ the word to denote simply a combination of pieces of material having definite relative motions, one of the pieces being regarded as fixed in space. Such a mechanism often represents kinematically some actual machine which has the same number of parts as the mechanism with the same relative motions. The essential difference is that in the case of a machine such parts have

to transmit or transform energy, and are proportioned and formed for this end, while in a mechanism the relative motion of the parts only is considered. We may look upon a mechanism, then, as being the ideal or kinematic form of a machine, and our work will be much simplified in most cases if we consider for kinematic purposes the mechanism instead of the machine. Such a substitution is also of the greatest service in the comparison and classification of machines; we shall find in this way that machines, at first sight quite distinct, are really related, inasmuch as their representative mechanisms consist of the same number of parts having similar relative motions, and only differing because a different piece is considered to be fixed in each case.

2. Constrained Motion.—On further consideration of the nature of a Machine as defined above, it will be noted that each part of the machine must have certain definite motions relatively to any other part, such definite motions being repeated again and again during the working of the machine. Thus the motion of a machine-part must be completely *constrained*, that is, the part must be free to move only in the manner desired to produce the required transformation of energy, and for it other unnecessary motions must be rendered impossible. Constrained motion of a body takes place when every point in the body is made to describe some definite and prescribed path. This constraint is effected in general by so forming and connecting the parts that all forces tending to disturb their constrained motion are balanced by stresses set up in the parts themselves. It is assumed, of course, that the machine remains uninjured by such stresses.

3. Pairs of Elements.—The nature of the connection between the parts of a machine will be best understood by taking a simple case and discussing the way in which some form of constrained relative motion of two bodies may be obtained. Suppose, for example, that a piece of material,

which we may call a, has to be capable of a motion of translation along a straight line, with reference to another piece, b, and is to have no other relative motion whatever. This must be accomplished by giving these pieces suitable forms. Such an arrangement as that sketched in Fig. 1

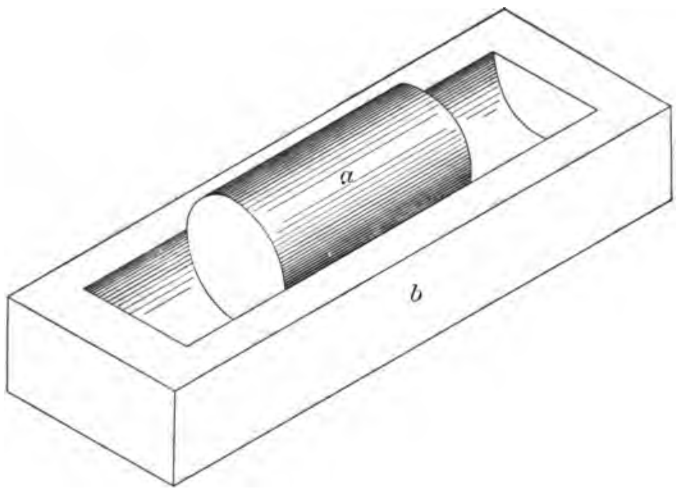

Fig. 1.

would not meet the case, for, although a executes the required movement so long as it remains in the groove formed in b and does not rotate on its axis in the groove, the forms shown do not prevent a leaving the groove in b or rotating in that groove.

It will be found that to attain the desired object some such forms as shown in Fig. 2 must be adopted, and that if this is done, the only possible motion of a relatively to b is that of simple translation along a straight line parallel to the edge of the groove or slot in b. The figure will recall to the reader the appearance of a steam-engine cross-head and its guides, a pair of bodies which have indeed the same relative motion as that described above.

We shall refer to a pair of bodies so formed as to permit

of partly or wholly constrained relative motion while in contact as a *pair of elements*, the elements being really the surfaces of contact, or working surfaces, of the pair of bodies. Such pairs are distinguished as being (*a*) *higher pairs* and (*b*) *lower pairs*. Lower pairs may be defined as those in which "the forms of the elements are geometrically identical, the one being solid or full and the other hollow or open" (Reuleaux). This definition involves the idea of surface contact to produce the required partial or complete constraint, while in the case of higher pairs constraint is produced by contact at a sufficient number of lines or

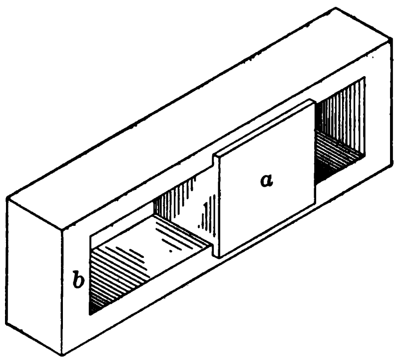

FIG. 2.

points. Mechanically, lower pairing in machinery is preferable, wherever possible. The reason for this is that wear takes place much more rapidly in a case where line or point contact occurs than in the case where surfaces of considerable extent are touching, other conditions being the same.

A pair of elements whose relative motion is completely constrained is said to be *closed*. Thus such a pair as is shown in Fig. 1 is not closed, while that of Fig. 2 is completely closed; for, as has been already pointed out, the only possible relative motion is one of pure translation in a straight line.

The nature of the relative motion of two bodies can only be defined when one of them is considered as being fixed. In the case of a pair of elements *ab*, *a* being fixed while *b* moves, we may have the same relative motion of *a* and *b* as when *b* is fixed while *a* moves, but the pair is said to be inverted, that is, the second element is fixed instead of the first. Examples of such *inversion* of pairs frequently occur in considering actual machines, and it is important to remember that, while inversion of a pair may cause no alteration of the *relative* motion of the elements themselves, it may, and generally does, alter their motion relatively to other bodies.

4. **Links and Chains.**—In studying any simple mechanism or machine, we find that each piece of material carries, or has formed upon it, one element of each of two or more pairs. Take for example the cross-head of a steam-engine; in addition to the surface which pairs with the guide bar or bars, the block has a cylindrical surface pairing with a similar one on the small end of the connecting-rod, and it thus carries, or links together, two elements belonging to two different pairs.

In general, then, a part of a machine forms a *kinematic link* connecting two or more elements, belonging respectively to two or more pairs, and the whole arrangement or combination of such links is known as a *kinematic chain*. This may or may not have such kinematic properties as to make it available as a mechanism; for we can easily imagine a kinematic chain which does not comply with our definition of a mechanism when one link is fixed. Consider the case of a linkwork formed of five bars, *a b c d e*, jointed at the angles as shown in Fig. 3. Suppose *a* to be fixed, then the motion of *c* or *d* relatively to *a* is not constrained, and such a chain, therefore, is not a mechanism as we have defined it.

It is most important to note, with regard to this point, that the motion of *c* with respect to *b* *is* constrained, i.e., *c* can only have one motion with regard to *b*, that of turning about the axis of the joint connecting them, whereas with

respect to a, c can be made to move in any number of different ways, depending in this case on the force or forces applied to the different bars. The motion of c with respect to a is therefore not constrained. This fact is illustrated in Fig. 3, where it is seen that if the links b and e take up the positions b' and e', c and d may be either at c' and d', or at c'' and d''. Such a kinematic chain as this is said not to be *closed*, and we define a *closed chain* as a series of links

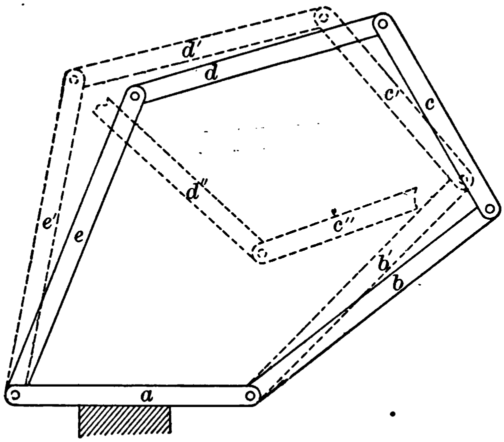

FIG. 3.

so connected that each of them has only one definite motion relatively to any other link. Thus if one link be fixed, the motion of any other can be determined. A closed chain having one link fixed is then equivalent to a mechanism.

The various ways in which closure is obtained in pairs and in chains will be discussed later.

A chain of which each link carries two elements is termed a *simple chain*, for a link cannot have a less number of elements than two. If, however, any link or links have three or more elements respectively belonging to three or more pairs, the chain is said to be *compound*. In some ways compound chains present more difficulties than do simple chains, but the kinematics of both kinds may be

studied by exactly the same methods. Fig. 4 shows a closed compound chain, which has been suggested as a straight-line motion. It will be seen that the link *a* is fixed, and that *b* carries one element of each of the pairs

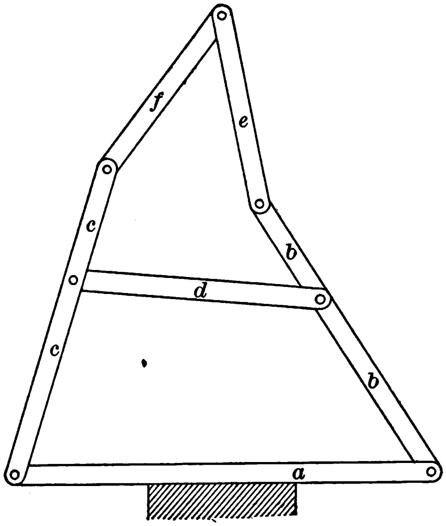

FIG. 4.

ba, *bd*, *be*, while *c* has upon it one element of each of the pairs *ca*, *cd*, *cf*.

It is worth while noticing that if the link *d* were removed the chain would no longer be a closed one. The particular mechanism shown in Fig. 4 will be again referred to.*

In the last two figures the links have been represented by straight bars. From a kinematic point of view, however, the mechanisms or chains would have been unchanged if the form of the bars had been altered in any way, always supposing that the axes of the joints remain parallel and at the same distance apart, and that the forms of the links are not such as to cause fouling or interference while the mechanism is in motion. It is evident that these remarks apply generally, and we may say that, as a rule,

* See Fig. 59.

the form or shape of a link in a chain is not of importance in kinematics, so long as the form adopted does not render impossible any portion of the required movement of the link. Questions of form and shape fall within the province of the science of Machine Design.

We have already seen that in discussing whether a kinematic chain is or is not equivalent to a mechanism, we suppose one link to be fixed, and we then proceed to determine whether the chain is closed or not; a closed chain having one link fixed being regarded as a mechanism.

The choice of the fixed link is left open, and by selecting different links of a kinematic chain different mechanisms are generally obtained. Thus, in general, from a given kinematic chain we may derive as many mechanisms as the chain has links. These mechanisms are called the *inversions* of the original chain, and, as in the case of the inversion of pairs, the exchange of one fixed link for another is known as the *inversion of the chain*. Many examples of such inversion will be met with in the following chapters.

5. **Motion and Position in a Plane.**—Kinematics is simply the science of pure motion, as is indeed indicated by its name (from *κίνημα*, motion), first suggested by Ampère. Some of the simpler propositions of pure kinematics will be given here before explaining their application in the special case of the kinematics of machines. They are based on geometrical principles, since they deal with the ideas of position and space. But it will be at once seen that the introduction of the ideas of time, and consequently of velocity and acceleration, extends the scope of the science of kinematics considerably beyond the limits of pure geometry.

Two chief classes of problems arise, the first dealing with the position and motion of a particle, and the second treating of similar questions relating to rigid bodies. The motion of non-rigid bodies is of course of a far more complex nature, and only a few elementary cases will fall within the limits of this work. Indeed the motion of such bodies cannot be

investigated apart from the forces acting on them, and its consideration falls within the province of Kinetics, rather than within that of Kinematics.

Motion is defined as change of position, and is known if the position of the point or body considered is known for every instant. The position of a point or of a body can only be defined in relation to another point or body (as the case may be) whose position is fixed, or, in other words, whose change of position is neglected. Position (and therefore motion) is then purely relative. When we speak of a mountain being ten thousand feet in height, we are referring the position of its summit to an arbitrary datum surface, that of mean sea-level. In stating the position of a point or body (a body being equivalent to a system of points) we must then refer to some other point or body, and in considering the motion of a point or system of points, such motion can only be imagined with reference to a second point or system of points, supposed to be fixed.

In the case of plane motion, this reference system is usually taken to be the surface on which is drawn the diagram representing the motion of the body considered. In order to define the plane motion of a plane figure, with regard to a plane, it is sufficient to know the motion of any two points in the figure with reference to the plane. The truth of this will be seen by considering that if the motion of one point only were known, we should still be ignorant of any rotation the figure might have about an axis perpendicular to the plane. The knowledge of another point's motion, however, defines such rotation.

In most cases, problems arising in the kinematic study of machines are found to involve the consideration of *Plane Motion* only.

A rigid body having Plane Motion moves in such a way that all planes originally parallel to a certain fixed plane (that of motion) remain parallel thereto during the whole movement of the body, while any point whatever in the body

moves in a plane either parallel to or coincident with the plane of motion.

A body moving in this manner will in fact have no motion of translation in a direction normal to the plane of motion, and the position of the body with respect to the plane of motion will agree exactly with the position of its projection on the plane of motion. Hence in considering the plane motion of rigid bodies, we need deal only with the kinematics of plane figures, and all propositions relating to the plane motion of plane figures will be applicable to that of rigid bodies.

It is not, in general, so necessary to trace out the whole motion of a body as to know what is its *instantaneous motion* at some given stage of its movement. By this term is meant the change of position executed by the body in a very small period of time. The manner in which these small changes of position follow one another must now be considered for the case of plane motion.

In Figure 5, let AB, $A'B'$, represent two successive positions of a plane figure (as defined by the position of two

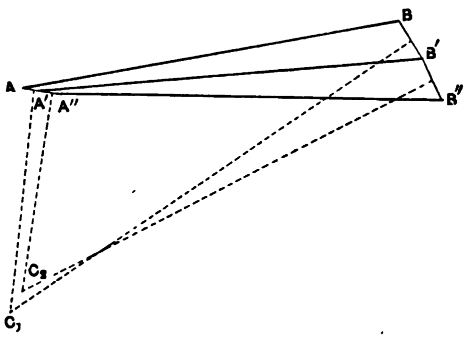

FIG. 5.

points A and B in it) at the beginning and end of an interval of time which is very small as compared with the whole period of motion.

Join AA', BB', and bisect the lines AA', BB', by straight lines perpendicular to AA', BB', and intersecting at C_1.

Then it is plain that $C_1A = C_1A'$ and $C_1B = C_1B'$, and if the point A had described a very small circular arc with centre C_1, its new position would have been A', and its path would have been indistinguishable from the line AA'. The actual infinitesimally small change of position of the point A is therefore the same as if it had been rotated in the plane of motion around an axis perpendicular to the plane and passing through C_1, and similarly for the point B. Thus, knowing the change of position of two points in the rigid figure considered, we say that the actual instantaneous motion of the body AB has been equivalent to a *virtual rotation* about the centre C_1. During the next instant the instantaneous motion may be around some other point C_2 indefinitely near to C_1, and so on. The point C_2 corresponds to the movement from $A'B'$ to $A''B''$. Thus to every part of the motion of AB, with regard to the plane, there corresponds a certain point C in the plane, about which an equivalent virtual rotation has taken place. Such points, as $C_1, C_2 \ldots$, are called the *instantaneous* or *virtual centres* of AB with regard to the plane. The locus of C_1, or the curve described by the point C on the plane, is known as the *centrode* of AB with regard to the plane, and, in general, it forms a continuous curve.

In the case of a rigid body having plane motion, it would be more correct to consider the equivalent rotation as taking place about a *virtual axis* (perpendicular to the plane of motion) of which the points $C_1, C_2 \ldots$ are the successive traces on the plane of motion. Such a virtual axis would then describe a surface in space, this surface being known as the *axode* of the body with regard to the plane of motion. For most cases of plane motion, however, we are content to simplify matters by considering the centrode instead of the axode. We shall see later that in more complex forms of motion the axode becomes of great kinematic importance. It is in every case what is called a ruled surface, i.e., a surface described by successive positions of a straight line in space.

Referring again to the plane motion of the figure AB (Fig. 5), let us inquire what happens if our construction fails. This will occur if the bisectors of the lines AA' and BB' are parallel, in which case the successive positions of AB are also parallel to one another, and the motion of the body, or of the figure it represents, is one of simple translation in a straight line. The virtual centre for such motion as this is then at an infinite distance, and we may regard any plane motion of translation in a straight line as equivalent to a rotation about an infinitely distant centre. Again, suppose that one of our reference points A does not change its position at all. It is easily seen that AB has now simple rotation about A, and during the continuation of this motion we have no longer a virtual but a permanent centre. It may happen that the lines bisecting AA' and BB' are coincident. A little consideration will show that in this case, since the triangles ABC_1 and $A'B'C_1$ must be equal in all respects, the point C_1 is at the intersection of AB and $A'B'$, produced if necessary; as before, a simple rotation about C_1 would suffice to move AB into the new position $A'B'$.

It is thus shown that in every case *the motion of a plane figure in a plane may be regarded as equivalent to a simple rotation about some actual or virtual centre*, whose position in the plane will be fixed in the case of simple rotation, or will be at an infinite distance in the case of simple translation. Such a virtual centre, however, is in general neither fixed, nor at an infinite distance, but changes its position as the body moves, and its locus in the plane is the centrode of the body with reference to the plane. Note that only rigid bodies or figures can have centrodes, for we assume that the position of our reference-line AB in the figure or body remains unchanged throughout the motion, and we represent a rigid body by the line joining the two points in question.

It has thus been seen that the centrode of a body with regard to the plane of motion is a curve described on that plane by the virtual centre of the body. Let us now con-

sider the relative motion of two bodies in a plane. Instead of supposing that the virtual centre M (Fig. 6) of the first body AB traces its centrode on the plane of motion, imagine that the curve is marked on a sheet of paper or surface rigidly attached to the second body CD, and that the body CD is fixed. The point M is then the one point common to the two bodies AB and CD *at which there is no relative motion*, for the only possible relative motion would be rotation about the point M, a motion which is non-existent as far as a point is concerned. M is the virtual centre of AB relatively to CD, but evidently it might equally well be called the virtual centre of CD relatively to AB. Next suppose that AB is fixed, and let CD have exactly the same relative motion as before. At the instant when the relative positions of AB and CD are the same as those just considered, the virtual centre will be the same point M, but it may now be supposed to describe its centrode on the body AB, and not on CD. This centrode (that of CD relatively to AB) will not be the same curve as that described before, although they must have one point M in common at any instant. It is evident, therefore, that the two centrodes corresponding to the relative motion of two bodies always touch at a point, which is the virtual centre for the instant considered, and we may represent such relative motion by the rolling on one another of a pair of centrodes. Further, we shall find that from the form of these centrodes we can determine the relative motion of the two bodies.

To make this clearer, the two cases of motion are represented in Fig. 6. AB and CD represent the original positions of the two bodies, and, CD remaining fixed, A_1B_1, $A_2B_2 \ldots A_5B_5$ represent successive positions of AB, the motion from AB to A_1B_1 corresponding to a rotation about a virtual centre M_1, and so on. The curve $M_1M_2 \ldots M_5$ is then the centrode of AB with regard to CD.

Next we have plotted the positions C_1D_1, $C_2D_2 \ldots C_5D_5$ which CD would occupy, supposing that the relative motion

were the same as before, but that AB now remained fixed. For example, in the figure C_3D_3 has the same position relative to AB that A_3B_3 has to CD, and so on for all the other positions. We now find the series of virtual centres M_1,

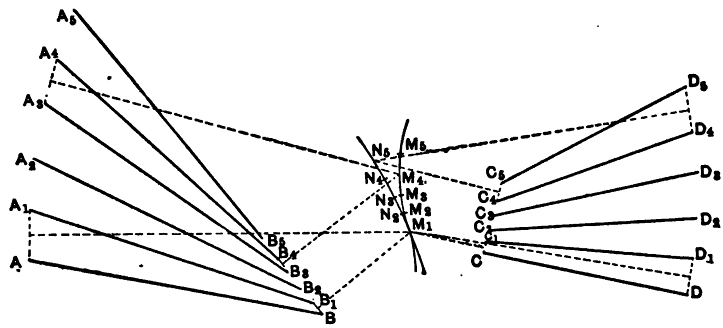

FIG. 6.

$N_2 \ldots N_5$ by the construction previously explained, and see that these centres lie on another curve touching the first at M_1 and forming the centrode of CD with regard to AB.

Remembering that this curve is attached to, or rather described on, the body represented by AB, suppose that CD remains fixed, while AB (with the centrode attached) moves from AB to A_1B_1, i.e., AB rotates instantaneously about M_1. If the movement is imitated by tracing AB and the curve M_1, $N_2 \ldots N_5$ on paper and placing AB in the position A_2B_2, it will be found that N_2 coincides with M_2. When AB is at A_3B_3, M_3 and N_3 coincide, and so on. Such successive coincidences can only occur if the curve M_1N_5 rolls on the curve M_1M_5.

In the same way if we trace CD and the curve M_1M_5, and let CD occupy its successive positions, we find that the points coincide as before, the curve M_1M_5 now rolling on M_1N_5.

Thus the given relative motion of AB and CD, through the successive positions shown on the figure, is represented by the rolling on one another of two curves, the pair of centrodes of the two bodies.

The reader is strongly recommended to satisfy himself of the correctness of the above statements by actually drawing a pair of bodies, and their centrodes for a given case of relative motion. Great care and accuracy in drawing are necessary in order to obtain correct positions for the virtual centres.

We have now discussed the case of the relative motion of two bodies in a plane, and have seen that their virtual centre describes a pair of curves, namely the centrodes, each being traced on one of the two bodies.

Suppose next that we have three bodies, represented, as before, by plane figures, and having any kind of relative plane motion. The three bodies will evidently have three virtual centres, while four bodies would have six, and so on; in fact, a kinematic chain having plane motion and consisting of n links will have $\frac{n(n-1)}{2}$ virtual centres connected with it, for it will easily be seen that the number of virtual centres must be that of the combinations of n things taken two at a time.

On examination of any particular case we shall see that the various virtual centres in a mechanism having plane motion are arranged in threes, each three lying in a straight line, whatever be the position of the mechanism.

The proof of this statement is as follows: Consider any three of the bodies, or links forming the kinematic chain or mechanism, and let us call them a, b, c. Denoting the virtual centre of a with regard to b by O_{ab} and remembering that this is the same point as the virtual centre of b with regard to a, we have for the three bodies considered the three virtual centres O_{ab}, O_{ac}, O_{bc}. First consider b as being fixed. Then with regard to the point O_{ab} any point in a has a simple motion of rotation, so that, for example, the point O_{ac} is moving instantaneously and relatively to b in a direction at right angles to the line $O_{ac} \ldots O_{bc}$.

Again, with regard to the point O_{bc}, any point in c, such

as O_{ac}, must be moving instantaneously and relatively to b in a direction at right angles to the line $O_{ac} \ldots O_{bc}$.

Thus the point O_{ac}, regarded as a point in a, is moving in a line perpendicular to $O_{ac} \ldots O_{ab}$; while if regarded as a point in c, O_{ac} moves in a line perpendicular to $O_{ac} \ldots O_{bc}$, b being regarded as fixed in each case. O_{ac} cannot have two separate directions of instantaneous movement at the same instant, hence the lines $O_{ac} \ldots O_{ab}$ and $O_{ac} \ldots O_{bc}$ are both perpendicular to the same line. They cannot be

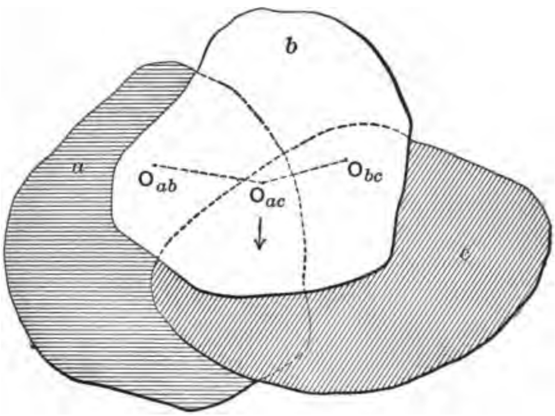

FIG. 7.

parallel, since they both pass through O_{ac}, and they therefore coincide in direction, i.e., the points O_{ab}, O_{ac}, O_{bc} lie on one straight line.

The position of the virtual centres in various mechanisms will be studied when we consider the relative velocities of their different parts. In many instances the proposition just given is of great assistance in determining the positions of the virtual centres in a mechanism.

6. Non-plane Motion.—In the majority of cases it will be found that the relative motions of the parts of machines are plane motions, either of rotation or translation, or both combined. Such motions can be studied geometrically by

the method indicated in the preceding section. It is possible (as will be seen later) to have a lower pair, in which the motion is non-plane. A somewhat limited number of cases of higher pairing also occur in which the motion is non-plane.

In every instance, however, in a closed pair, we have seen that there must be continuous contact of the surfaces, and it follows that the most general possible relative motion of two parts of a mechanism is represented by the motion of one rigid body continuously touching another at a point or series of points.

Any such motion must be of the nature of sliding, rolling, or spinning, separately or combined.

Simple rolling takes place if the instantaneous axis lies in the common tangent plane at the point of instantaneous contact.

Simple spinning exists when the instantaneous axis is the common normal at the point of contact.

Suppose that the relative motion is such that the instantaneous axis passes through the point of contact, and is neither in nor perpendicular to the tangent plane. The motion is then combined rolling and spinning. If the instantaneous axis does not pass through the point of contact, the rolling and spinning will further be combined with a sliding motion.

We have a familiar example of combined rolling and sliding in the mutual action of a pair of teeth in an ordinary spur-wheel; the motion of the balls in a bicycle bearing, again, is a case of combined rolling and spinning.

The links of a certain class of mechanism are found to have such motions that their instantaneous axes all pass through a fixed point, while each portion of every link remains at its own constant distance from that point. Such motion is called *spheric motion*, because any given point on a link must be always on the surface of a sphere described about the fixed point as centre. It is evident that

the most general case of spheric motion is that of a rigid body of which one point is fixed, and any kind of spheric motion can be made up by combining spins about axes passing through the fixed point. Plane motion may be looked upon as a particular case of spheric motion, in which the radius of the spheres is infinitely large.

7. **Freedom and Constraint.** — We have seen that the essential feature of a kinematic pair is the mutual constraint due to the forms of the two elements of which the pair is composed. Before considering the ways in which constraint or closure is actually applied it will be well to examine briefly the conditions on which the freedom of movement of a rigid body depends.

The most general motion of a free rigid body may be looked upon as being a combination of three independent rotations about three rectangular axes, with three independent motions of translation along those axes. Such a body may then be said to have *six degrees of freedom*, one of which is taken away (or one degree of constraint is imposed) when any one of these six modes of movement is rendered impossible. Suppose that the free rigid body is forced to touch a smooth fixed surface at one point, one degree of freedom is lost, for no translation can take place in a direction normal to the tangent plane to the surface at the points of contact. The three motions of rotation, however, still remain possible, and so does motion of translation in any direction parallel to the tangent plane at the point of contact. A second point of restraint may be arranged so as to prevent one motion of rotation, or a second motion of translation, according to its position with regard to the first point of restraint and with regard to the form of the body. A third point of restraint causes the body to lose a third degree of freedom, and, finally, it will be found that all six degrees of freedom are lost, and the position of the body is fixed if six of its points are made to rest on six

portions of the surface of the smooth fixed body, and if these portions are properly formed and placed.*

It may be shown that in general six conditions are required to completely determine the position of a rigid body, or, expressing the same thing in another way, six coordinates specify the position of one rigid body relatively to another, considered to be fixed.

The definitions of a closed pair or of a closed chain given in §§ 2 and 3 thus mean that any element or link in a closed pair or chain may have only one degree of freedom as referred to the fixed element or link.

Consider, for example, a screw turning in a fixed nut, like the screw of a micrometer gauge. The position of such a screw is determined exactly if an arm attached to its head is forced to remain in contact with a fixed stop on the body of the gauge, and we say, therefore, that such a screw has only one degree of freedom, inasmuch as its position is fixed by one point of constraint. The motion of a screw in its nut, a motion of translation accompanied by a definite and proportional motion of rotation whose axis is the direction of translation, is the most general kind of motion that can be possessed by a body having only one degree of freedom.

The reader will notice that in two special cases, namely, when the pitch of the screw is infinite, and when the pitch is zero, the twisting motion of the nut becomes a mere translation or a mere rotation, both being specially important as plane motion involving one degree of freedom.

In a similar way such a body as the connecting-rod of a direct-acting steam-engine is said to have constrained motion, having only one degree of freedom. The only possible motion at any instant for a given point on the rod is that of rotation about a certain virtual axis parallel to the axis of the crank-shaft.

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* See § 60, Chapter VII.

Such a contrivance as a ball-and-socket joint cannot be regarded as a closed pair, for the ball has three degrees of freedom with regard to the socket. The ball has one point fixed, its centre, thus rendering all motion of translation impossible, and causing three degrees of constraint. The socket in fact might be replaced by three pairs of points touching the sphere at the ends of three diameters, each pair of points corresponding to one degree of constraint.

Further examples may easily be imagined: the method of determining the conditions as to freedom and constraint in any particular case will be evident from the instances just given.*

8. Elements and Pairs in Rigid Links.—It has been pointed out that the pairs of elements formed on the links of which a mechanism is made up are of two kinds, namely, *lower pairs*, in which the elements are in contact with each other over the whole or part of the area of certain surfaces, and *higher pairs*, in which such contact occurs only at certain points or along lines of points.

In those portions of machines which are rigid the elements must have forms which can be readily produced by the ordinary processes of the workshop. Accordingly we find that their shapes are such as can be formed either in the lathe or the milling-machine, or by one of the many machine tools in which the cutting-tool describes a straight line with reference to the work. The rigid elements forming the closed pairs in machines therefore have in general for their working surfaces either surfaces of revolution, plane surfaces, or screw surfaces.

From the definition of *Lower Pairs* it is also plain that the forms of their elements must be such as to fit one another not only in one position, but in any position they may take up during their relative motion. It is plain also that two

* The reader may refer to Thomson & Tait, Natural Philosophy, Part I, Sections 195–201; also Tait, Enc. Brit., art. Mechanics.

surfaces of revolution, the one full and the other hollow, will fulfil this condition, and if properly formed, so as to prevent any sliding along the axis of revolution, will constitute a closed lower pair in which either element can only have constrained motion relatively to the other. Such pairs are called shortly *turning pairs*, and Fig. 8 represents two bodies, *a* and *b*, so shaped as to form such a turning pair. The body *b* is partly cut away, to show more clearly the outline of *a*.

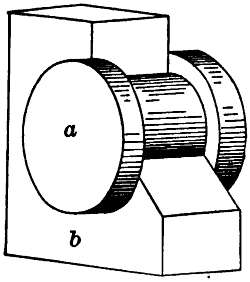


FIG. 8.

The same condition (of fitting each other in any position) obtains in the case of a screw of uniform pitch and its nut. The relative motion is also constrained, as has already been stated, and consists of a motion of rotation around the axis of the screw, combined in a constant ratio with a motion of translation along that axis. Such a pair of screw surfaces forms a *screw-pair.**

In general a lower pair formed by two cylindrical or prismatic surfaces will have constrained relative motion, because it will only be possible to give one body a motion of translation along the generating lines of the prism or cylinder relatively to the other body. If, however, the forms are circular cylinders, which are, of course, surfaces of revolution, then indefinite turning also is possible, the motion ceases to be constrained, and the pair is no longer closed. A pair of cylindrical or prismatic surfaces for which sliding only is possible is called a *sliding pair* (see Fig. 9).

On examination it will be found that pairs of conical and other forms of surfaces generated by straight lines do not fulfil the conditions of continuous fitting or contact during motion, unless they are at the same time surfaces

* See Chapter XI.

of revolution. Non-cylindrical ruled surfaces in machines therefore have usually to take part in *higher pairing*.

The three classes of lower pairs just discussed are then the only ones found in the rigid portions of machines. Examples of each kind will present themselves on examining a few simple machines, and the means of constraint should be noticed in each case. For instance, in a shaft-

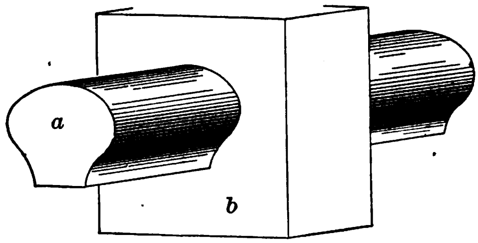


FIG. 9.

journal, endways motion or sliding of the shaft in its bearing is prevented either by making the diameter of the journal smaller than that of the adjoining portions of the shaft, or by securing collars on either side of the bearing.

All the forms of ruled surfaces mentioned above, and occasionally plane surfaces, surfaces of revolution, or screw surfaces, are found as portions of higher pairs, as well as of lower pairs.

A simple arrangement of higher pairing can frequently be used to give motion of a kind which could only be otherwise obtained by a complex chain of lower pairs.

It is important to notice that higher pairs give relative motion of a much more complex kind than is attainable by the use of lower pairing. This fact is pointed out by Burmester,* and is expressed if we say that supposing *a*

* Lehrbuch der Kinematik, §§ 114, 116.

and b are two elements of a closed pair, and if a point A in a describes the same curve on b as a point B in b (originally coinciding with A) describes on a, then the pair is a lower pair. If, on the other hand, A describes on b a line or curve different from that described by B on a, we have a case of higher pairing. Thus in the case of a lower pair no alteration of the relative motion occurs whether we consider one or the other of the elements as being the fixed one.

Lower pairing is the more important from a constructive point of view, because the elements of a lower pair have a simpler relative motion, they are able to resist wear when transmitting heavy loads, and they can easily be made tight under fluid pressure. These are properties not possessed by higher pairs.

9. Pairing of Non-rigid Links.—Passing on to the pairing of *non-rigid links* in mechanisms, it is found that these links may be classed under the following heads:

(1) Flexible bodies, such as ropes, belts, or chains. These are almost invariably paired with cylindrical surfaces on to or from which they unwrap or wrap themselves. Such pairing may be called *tension pairing*, since the rope, belt, or chain is necessarily in tension.

(2) Pressure links, which continually exert pressure on the elements with which they pair. These links generally consist of portions of fluid, such as air, steam, or water, and pair with the interior of the vessels containing them. Such a pair is known as a *pressure pair*.

Springs often form most important portions of mechanisms and machines. They may be arranged so as to be in tension or in compression, and the resulting pairs may be said to be tension or pressure pairs, as the case may be.

Actually all machine parts are elastic and so act to a certain minute extent as springs, but in kinematics we neglect all small changes of form, and consider such pieces as being rigid, classing under the head of springs only those

portions of machines whose elastic deformations under load are considerable in extent when compared with the proper motions of the other machine parts or links with which they pair.

Non-rigid links will be considered at greater length subsequently.

10. Classification of Mechanisms. — In attempting to classify mechanisms, which are made up of various kinds of links and involve so many kinds of pairing, we are impressed with the magnitude and complexity of the task. It may be said, in fact, that up to the present no wholly satisfactory kind of machine classification has been proposed. Some account of what has been done in this direction will be found in Chapter XIII; for present purposes it will be sufficient to consider mechanisms under three heads.

(1) Those involving only plane motion. These may be called shortly *Plane Mechanisms*, and form by far the most mportant and numerous class.

(2) Mechanisms involving spheric motion, or, more briefly, *Spheric Mechanisms.*

(3) Chains the relative motion of whose links is neither plane nor spheric, but of greater complexity.

It is, however, to be understood that a mechanism of the third kind may contain certain links whose motion is plane or spheric, while any of them may include examples of both lower and higher pairing.

A well-known instance of a spheric mechanism is Hooke's joint, the characteristic property of such chains being that the axes of the turning pairs they contain meet in a point. In the third class the most common examples are screw mechanisms.

There is another method of classifying machines according to their geometrical properties, and according to the methods necessary for determining the various virtual centres of their links. Following this system, we should say that mechanisms of the *First Order* are those in which,

having given the relative position of *any two* links, the positions of all the other links may be found by geometrical construction of straight lines and circles. From this it follows that in such mechanisms, having given the whole mechanism in one position, we can find geometrically all its other possible positions, and the virtual centre of each link relatively to every other. Mechanisms not possessing these properties belong to higher orders, and are of comparatively infrequent occurrence.

CHAPTER II.

POSITION, VELOCITY, AND ACCELERATION.

11. Velocity.—While Kinematics in its general sense comprises all kinds of problems dealing with pure motion, the number of such problems falling within the province of the Kinematics of Machines is somewhat limited. We shall consider in this chapter some elementary notions concerning velocity which are applicable to the purposes of the Kinematics of Machines. Methods of studying the position and motion of a point or rigid body from a geometrical point of view have already been indicated; it now remains to investigate not only the amount by which such position is changed during motion, but the rate of such change of position. Going a step farther still, it may be asked, does such velocity increase, diminish, or change in any way as time goes on, and if so, at what rate?

The rate of change of position of a point or body is called its *velocity*. A body, as we have seen, may change its position by a motion of translation, or by one of rotation. Hence we distinguish between linear and angular velocity. The former is measured by the space passed over in unit of time, and is usually expressed in feet per second, although other units, such as miles per hour or knots, are adopted in special cases. The latter is measured by the angle described in unit of time, the natural unit being therefore one radian per second. Engineers, however, commonly measure angular velocity in revolutions per minute. Either kind of velocity may be uniform or variable.

It is important to note that the term velocity involves

the ideas of both speed, direction, and sense. In other words, a velocity is a vector quantity, and, like other vector quantities, may be represented by a straight line of definite length, this length being proportional to the speed, or magnitude of the velocity, measured in feet per second, radians per second, or whatever units are to be employed.

In the case of linear velocity the direction of the vector or straight line representing the velocity on the diagram is taken to represent the *direction* of the motion. Thus, for example, we might draw upon a map a line running east and west, and 2 inches in length, and take this line as representing a linear velocity of 2 miles per hour, or 2 feet per second, either from east to west, or from west to east. The *sense* of the motion may be either from east to west, or from west to east. In order to indicate the sense, we place upon the line a small arrow-head so as to show the point towards which the body is moving (see Fig. 12).

In the case of angular velocity the direction of the vector on the diagram would be taken to represent the direction in space of the axis about which the spin or rotation is taking place, and a line similar to that mentioned above would mean a spin of two radians per second, or two revolutions per minute, according to the scale, about an axis lying east and west. This rotation may be either right-handed or left-handed, and it is therefore customary to indicate the sense by placing the arrow-head in such a fashion that the spin will appear to be right-handed, or clockwise, when looking along the axis and following the arrow-head.

It is plain that in this manner a velocity, whether linear or angular, may be completely represented by a vector, having magnitude, direction, and sense.

12. Uniform Velocity.—A body having uniform velocity (whether angular or linear) performs equal changes of position in equal times. If the body has a uniform linear velocity v, it describes a distance vt in time t, where t is any number of units of time. Calling s the space described, we have

therefore $s = vt$. Similarly, if the uniform velocity is angular and is denoted by ω, any line on the body in a plane perpendicular to the axis of rotation describes ω radians in each second and therefore ωt radians in t seconds. Hence, calling θ the angle described in t seconds, we have

$$\theta = \omega t.$$

If a point, at distance r from the centre about which it moves in a circular path, has a linear velocity v, its angular velocity is measured by the angle subtended at the centre by the path it describes in one second. Hence

$$\omega = \frac{v}{r} \quad \text{or} \quad v = \omega r.$$

13. Variable Velocity. — In general a moving body varies its speed as well as its direction of motion. It is easy by observing the time taken to travel over a known distance, for example in a train, to calculate the *average speed* of the train during the interval considered. This does not tell us, however, the *actual speed of the train at any instant* during the interval of time, which may be quite different from the average speed.

The velocity at any instant, or instantaneous velocity, is measured by the space (or angle, as the case may be) which would have been described in a unit of time if the motion had continued uniformly, during that interval, at the same rate as at the instant considered. The word instant is here used to mean an indefinitely small interval of time.

We are not able to measure the distance (or angle) described during an indefinitely small interval of time, and therefore have to obtain the value of the instantaneous velocity of a body in another manner.

This will be best understood by a numerical example. Suppose that a man in a street-car at 12 o'clock finds that

in 10 seconds the car traverses a distance of 200 feet. This gives 20 feet per second as the average speed during the 10 seconds after 12 o'clock. Suppose that other observations taken during the first $1\frac{1}{5}$, 2, and 4 seconds showed that during these times the car travelled 30, 48, and 100 feet, corresponding to average speeds of 25, 24, and 21.75 feet per second. It is evident that the speed must really have been continually diminishing, and that the shorter the time during which the observation was made, the more nearly do we

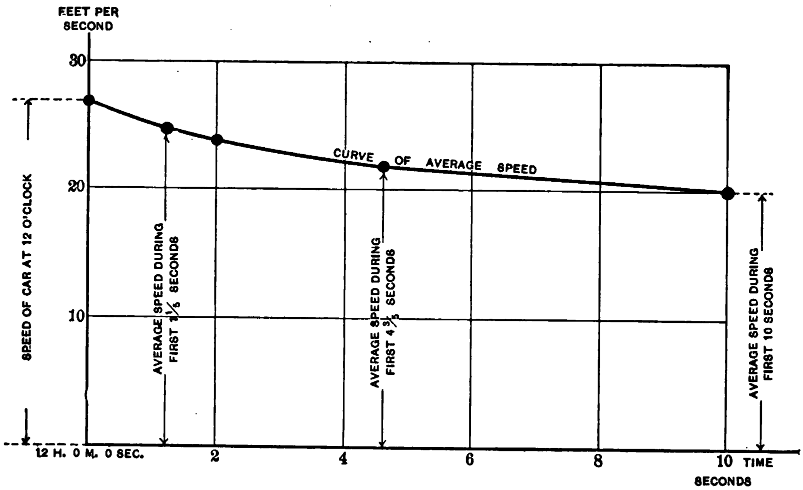


FIG. 10.

obtain the speed at which the car must have been travelling at 12 o'clock. To arrive at this more exactly, since we cannot measure the distance passed over in an infinitely small interval of time, we plot a curve from our observations, as in Fig. 10, and see that the speed at 12 o'clock must have been 27 feet per second. In mathematical language, if Δs be the distance traversed in a small interval of time Δt, the average velocity during that small interval is $\frac{\Delta s}{\Delta t}$, while the velocity at the instant beginning the interval is measured by

diminishing Δt indefinitely, and finding the limiting value of $\frac{\Delta s}{\Delta t}$, or, in the language of the calculus $\frac{ds}{dt}$. Thus

$$v = \frac{ds}{dt}.$$

The same reasoning, of course, applies in the case of angular velocity, where we should write

$$\omega = \frac{d\theta}{dt}.$$

Compare these with the corresponding expressions in the case of uniform velocity.

14. **Uniform Acceleration.**—A body moving with uniform acceleration changes its velocity by equal amounts in equal times. Thus suppose that in time t the velocity changes from v_1 to v_2; we have, if a is the acceleration,

$$a = \frac{v_2 - v_1}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, the average velocity during the time t is $\frac{v_2 + v_1}{2}$, the arithmetical mean between the initial and final velocities; hence if s be the space described,

$$s = \frac{v_2 + v_1}{2} t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From these two expressions we find

$$s = \frac{v_2^2 - v_1^2}{2a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But $v_2 = v_1 + at$. Substituting in (3), we get

$$s = v_1 t + \frac{a}{2} t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the case of angular velocity precisely similar relations hold, so that, calling α the uniform angular acceleration, ω the angular velocity at the beginning of the time t, and θ the angle described, we have, instead of (4),

$$\theta = \omega_1 t + \frac{\alpha}{2} t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

To express the velocity in terms of distance (or angle) and initial velocity we shall have instead of (3)

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

while the expression connecting velocity, acceleration, and time is

$$\omega_2 = \omega_1 + \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

As an example of the use of these expressions, suppose a wheel is revolving thirty times per second and comes to rest in 12 seconds. How many revolutions will it make in coming to rest if uniformly retarded?

We have $\omega_2 = \omega_1 + \alpha t$; hence

$$12\alpha + 30 \times 2\pi = 0,$$

and $\alpha = -\frac{60\pi}{12} = -15.71$ radians per second per second.

Again, $\omega_2^2 = \omega_1^2 + 2\alpha\theta$; hence

$$(60\pi)^2 - 10\pi\theta = 0,$$

and $$\theta = \frac{(60\pi)^2}{10\pi} = 360\pi \text{ radians}.$$

Hence the wheel comes to rest in 180 revolutions.

Again, a train starting from rest has a uniform acceleration of half a mile per hour per second. How far will it have travelled before attaining a speed of 30 miles per hour, and in what time will this occur?

In the equation (4) above we have $s = v_1t + \frac{1}{2}at^2$.

Here $v_1 = 0$, t evidently will be 60 seconds, and $a = \frac{2640}{3600} =$ 0.733 feet per second per second. Thus

$$s = \frac{0.733 \times 3600}{2} = 1218 \text{ feet.}$$

It should be noted that it is as incorrect to speak of an acceleration of so many *feet per second* as it would be to say that a body has a velocity of so many *feet*, without mentioning the unit of time.

15. Acceleration in General.—The determination of velocity and acceleration in the case of non-uniform or non-

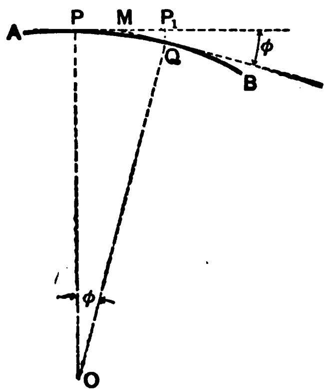


FIG. 11.

uniformly accelerated motion will be discussed later. Acceleration is defined generally as rate of change of velocity with regard to time, and so far we have used the term as meaning change in the *magnitude* of the velocity, whether

linear or angular. Strictly speaking, however, a change in the *direction* of motion in linear velocity, or in the position of the axis of rotation in angular velocity, is also an acceleration. In fact, a point travelling in a circular path around a fixed point has an acceleration impressed upon it, although its angular velocity may be uniform, and such acceleration is called *radial*, for reasons which will presently be seen.

In Fig. 11 let AB represent a portion of the curved path alng which a point is travelling with a linear velocity v, whose direction is continually changing. Let ρ be the radius of curvature OP of a very small portion PQ, and O the centre of curvature, φ being the very small angle between the tangents at P and Q, an angle so small that the arc PQ is not sensibly different from its chord.

Consider the acceleration in a direction parallel to PO. The time taken for the particle to travel from P to Q will be $\frac{PQ}{v}$. But during this time the distance traversed under acceleration a is P_1Q parallel to PO. Hence (if a is constant)

$$P_1Q = \tfrac{1}{2}a\left(\frac{PQ}{v}\right)^2 \text{ and } a = v^2\frac{2P_1Q}{(PQ)^2}.$$

It is known that for very small angles the numerical value of the sine of an angle is sensibly the same as the angle itself (of course expressed in circular measure). Also in the figure if we make φ small enough, $MQ = \tfrac{1}{2}PQ$, the error in this statement diminishing as φ diminishes. Hence *for an indefinitely small value of* φ we may say that

$$\frac{2P_1Q}{PQ^2} = \frac{P_1Q}{PQ.MQ} = \phi \times \frac{1}{PQ} = \frac{PQ}{OP} \times \frac{1}{PQ} = \frac{1}{\rho} \text{ and hence } a = \frac{v^2}{\rho}.$$

The statement is exactly correct, and a is the radial acceleration at P, because we have taken φ as being the angle described during an indefinitely small interval of time.

The earth's equatorial radius is 4000 miles, and the

earth makes one rotation on its axis in about 86,200 seconds. What is the radial acceleration of a particle on the earth's surface at the equator?

Linear velocity of a point at equator

$$=\frac{2\pi \times 4000 \times 5280}{86200} \text{ feet per second.}$$

Thus $$\frac{v^2}{\rho}=\frac{(2\pi)^2 \times 4000 \times 5280}{86200^2}$$

$$=0.112 \text{ feet per second per second.}$$

It is often necessary to find the acceleration of a body along its virtual radius; this is of course determined in exactly the same way as the radial acceleration with regard to a permanent centre.

16. Composition of Velocities and Accelerations.—It has been already pointed out that velocities, whether linear or angular, can be represented by straight lines of definite length, sense, and direction, and are in fact *vector quantities*, as distinguished from *scalar quantities*, such as mass, energy, and so on which have simply numerical values. Accelerations are also vector quantities.

The *resolved part* of a vector in any new direction is found by projecting its original length on the new direction.

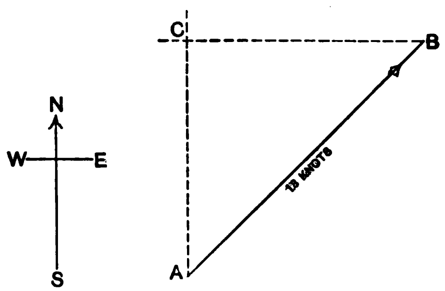


FIG. 12.

If, for example, a ship is proceeding northeast at a speed of 13 knots, represented by the vector AB (1 knot being a

speed of 6080 feet, or 1 nautical mile, per hour), its resolved velocity in a northerly direction is represented by

$$AC = AB \cos 45^\circ = 13 \times 0.707 = 9.19 \text{ knots.}$$

This shows that each hour the position of the ship is 9.19 nautical miles farther to the northward.

Again, suppose that the ship, still steering N.E. at the same speed, runs into a current whose speed is 4 knots due east, what will be the real velocity of the ship relatively to the earth? *Relatively to the water* its speed is still 13 knots

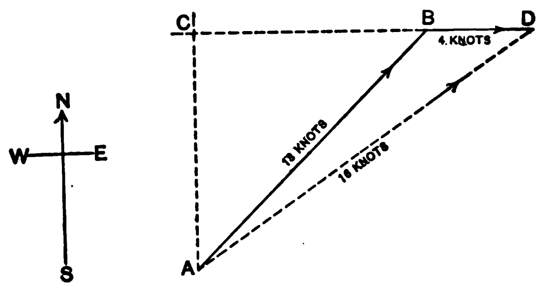


FIG. 13.

in a N.E. direction, but the water is itself moving, and at the end of the hour the ship will evidently be at D, a position obtained by measuring 4 nautical miles east from B. On calculation it will be found that at any time during the hour the ship has been moving relatively to the earth along the line AD, and its real speed over the ground (about 16 knots) will be measured by the length of AD, the third side of a triangle, whose other two sides represent respectively the velocity of the ship relatively to the water, and the velocity of the water relatively to the earth. We say, then, that the vector AD represents the *resultant* of the two vectors AB and BD, obtained by the process of vector addition.

The above example deals with plane motion in a straight line only. But if we are treating of the motion in space of a body having six degrees of freedom, its motion may be considered as made up of three motions of simple translation and three motions of rotation, which, when compounded according to the method just explained, constitute the actual motion of the body.

It must not be forgotten that the resultant of two or more angular velocities can be found in exactly the same way as for linear velocities. As already explained, it is customary to indicate an angular velocity by a vector (as in Fig. 14), representing the numerical value of the velocity by the length AB, the direction of the axis by the direction of AB, and the sense of rotation by drawing AB in such a manner that the rotation is clockwise, or right-handed, when looking from A to B.

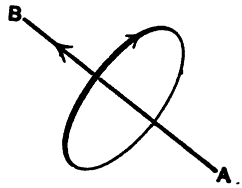


FIG. 14.

It is often necessary to compound or to resolve spins or angular velocities according to the method of vector addition, which will be already familiar to most readers under the name of the triangle of velocities, or the parallelogram law for the composition of vectors.

17. Resultant Acceleration.—In Fig. 15 let AB represent the original velocity of a particle, and suppose that accelerations represented by BC, BD are impressed upon the particle. Then BC and BD may be taken to represent the velocities generated in one second, corresponding respectively to the two accelerations.

If now the acceleration BC had alone acted on the point, its velocity at the end of one second would have been AC. Again, if AC had been the original velocity and an acceleration BD had been impressed, the final velocity at the end of one second would have been AD', where CD' is equal and parallel to BD. The two accelerations, therefore, have changed the original velocity from AB to AD'. But this

effect would have been produced by compounding with AB *for one second* a velocity BD', and we may therefore look on BD' as representing the change of velocity in one second, due to the action of the accelerations BC and BD. In

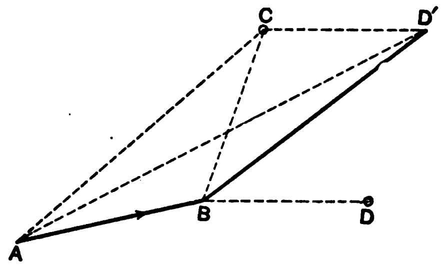


FIG. 15.

other words, BD' is the resultant of the accelerations BC and BD.

The general rule for the composition or addition of vectors, then, is that the resultant of two vectors is the diagonal of a parallelogram (or the third side of a triangle) of which the two components form the two adjacent sides. In this way we can find the resultant of any number of velocities, or of accelerations, either linear or angular. The same rules apply to the composition of any other vector quantities.

18. Diagrams of Displacement and Velocity.—In studying the motion of a body, whether linear or angular, it is necessary to know the position of the body at every instant during the motion, if we desire full information as to its velocity and acceleration. We have seen that if we only know the position of the body at certain times we can obtain the value of the average velocity between those times, but cannot tell exactly how the real velocity has changed.

It would of course be very cumbersome to have to state in words or figures a sufficient number of particulars to give us a practically complete knowledge of the position, velocity, and acceleration of a body, and therefore in such cases

graphic methods of representation and calculation are generally adopted.

For example, in order to determine the velocity of a body whose changes of position are known, such a diagram as Fig. 16 is constructed. Two axes, *OA*, *OB*, are drawn at right angles, and distances measured parallel to *OA* according to any convenient scale are considered to represent time,

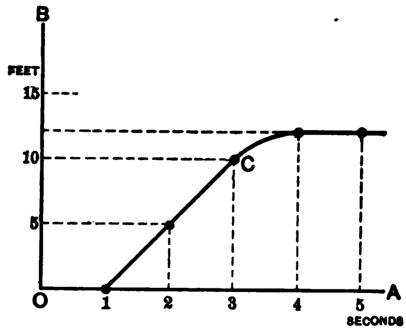


FIG. 16.

while lengths measured parallel to *OB* represent either the distance that has been traversed by the body, reckoning from some known position, or the angle turned through by the body at any given instant. This quantity we may call the displacement of the body, and it may be either linear or angular.

For instance, from the figure we see that after the lapse of 3 seconds the body in question has moved 10 feet from its original position, and we might give the information contained in the diagram in a less complete form in the shape of a table, thus:

Time........	0	1	2	3	4	5 seconds.
Displacement.	0	0	5	10	12	12 feet.

From the diagram, however, we are enabled to gather further particulars, for it is plain that the curve of displace-

ment during the second and third seconds is straight, i.e., distance is increasing proportionally to time, or the velocity is uniform and the speed 5 feet per second. During the fourth second the distance increases more and more slowly, and then remains constant, hence the velocity diminishes and finally ceases. Thus plainly, in order to find out the velocity at any instant from such a diagram, we have only to determine the rate at which distance (or angle) is increasing or diminishing at that instant. We shall see that this information can be obtained from the form of the curve.

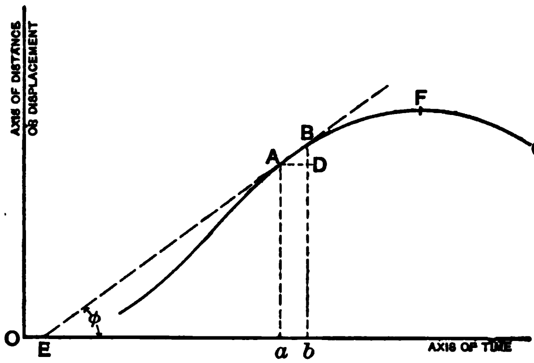


FIG. 17.

In Fig. 17 let ABC be any curve of displacement on a time base, and let Aa, Bb represent the distances (or angles) corresponding to the times Oa, Ob. We wish to determine the velocity at the point A, i.e., after the lapse of the time Oa.

It has already been pointed out that to find the instantaneous value of a velocity, or the velocity at any instant, we must take what is really the average velocity during an infinitely small time, or, if Δs be a small change in position and Δt the corresponding interval of time, the instantaneous velocity is the limiting value of $\frac{\Delta s}{\Delta t}$. In the figure let A and B be very close together; evidently $BD = Bb - Aa = \Delta s$, the

change in distance during the time Δt. The small interval of time Δt is represented by the length $ab = AD$. Hence the ratio

$$\frac{BD}{AD} = \frac{\Delta s}{\Delta t}.$$

Draw the straight line AB and produce it to cut the axis of time in E, making an angle $AEa = \varphi$. Then $\frac{\Delta s}{\Delta t} = \frac{BD}{AD} = \frac{Aa}{Ea} = \tan \varphi$. Now suppose we diminish Δt, making B approach A more and more nearly, as is shown on a larger scale in

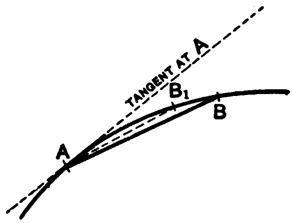


FIG. 18.

Fig. 18. The chord AB becomes AB_1 and, as Δt diminishes, approaches more and more closely to the tangent to the curve at A, and, if Δt is made infinitely small, AB will coincide with that tangent. Still, however, the value of $\frac{\Delta s}{\Delta t}$ is shown by the numerical value of $\tan \varphi$, and in the limit

$$\text{velocity} = \tan \varphi = \frac{ds}{dt}.*$$

Accordingly we may say that to find the velocity of a body at any instant from its diagram of displacement *drawn on a time base*, we have only to draw at the point corresponding to that instant a tangent to the curve. The slope of that tangent, as measured by the tangent of the angle it makes with the axis of time, is proportional to the velocity, and indeed represents the velocity numerically if the distances are measured to their proper scales. For example, at C in Fig. 16, to the scales marked, $\tan \varphi$ has the value $\frac{10 \text{ feet}}{2 \text{ sec.}}$ and shows, therefore, that at 3 seconds the body in

---

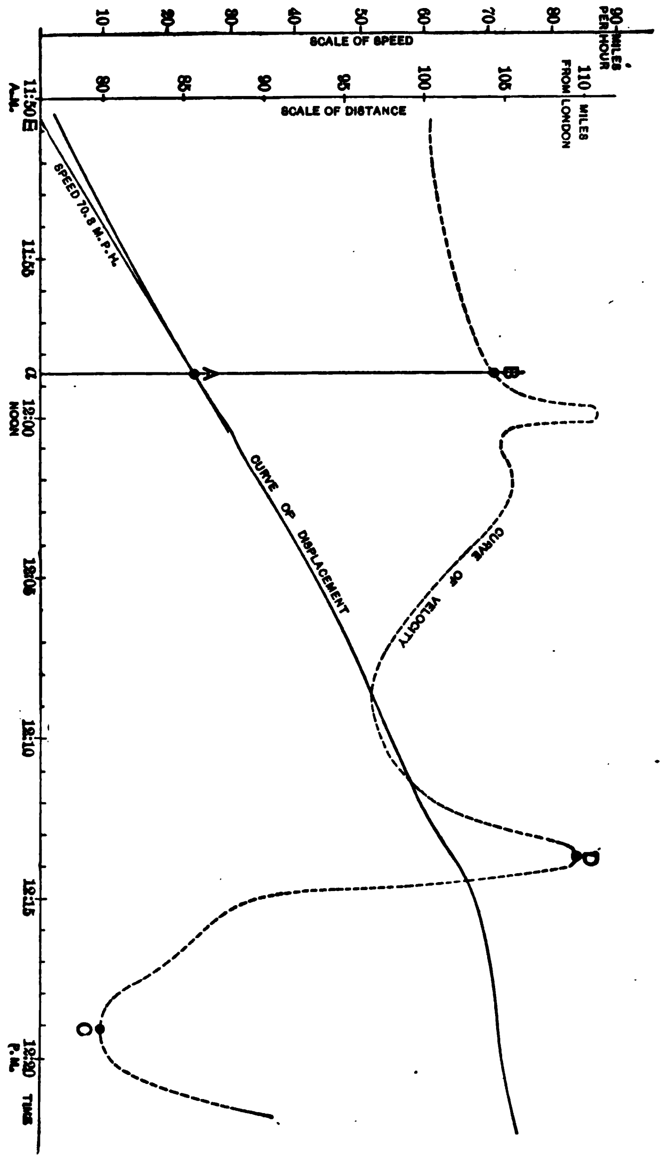
* See § 13.

question had a velocity of 5 feet per second. If a scale of miles had been marked along the axis of distance, while hours had been measured along the axis of time, our velocity would have been read in miles per hour.

The reader should notice that at the point F in Fig. 17 the displacement has ceased to increase with increase of time, and is about to decrease; the body has in fact reached its maximum distance from its starting-point. At this point the body must of course cease to be moving for an instant, which is shown by the fact that the tangent to the curve at F is horizontal, corresponding to zero velocity. After F the velocity will of course have to be reckoned negative, since distance is now diminishing as time goes on.

If the velocity at every instant could be measured in this way and a new curve drawn on a time base, having ordinates at each instant proportional in length to the velocity at that instant, we should obtain a curve or *diagram of velocity.* Actually we obtain only a sufficient number of values to give us a series of points on the curve, through which the curve can be drawn. This has been done as an example in Fig. 19. The full curve is a diagram showing the distance from London at times between 11.50 A.M. and 12.20 P.M., on a certain date, of the London and Exeter express on the Great Western Railway, the times of passing various stations and mile-posts having been carefully noted. From this full curve has been drawn a dotted one, the height of which at any point is proportional to the speed of the train at that time; this dotted line in fact is a curve of velocity.

It will be seen that the curve of displacement slopes continually upward, showing that the train did not stop during the time considered. Its speed, however, was very variable. The train performed the whole journey from London to Exeter, 194 miles, in 3 hours 38 minutes without a stop; thus the average speed was 53.4 miles per hour. At about 12.14, however, the speed for a very short time is seen to have exceeded 80 miles per hour, as shown at D, while



10
20
30
40
50
60
70
80
90 MILES PER HOUR
SCALE OF SPEED
80
85
90
95
100
105
110 MILES FROM LONDON
SCALE OF DISTANCE
SPEED 70.8 M.P.H.
CURVE OF DISPLACEMENT
CURVE OF VELOCITY
A
B
C
D
E
a
11:50 A.M.
11:55
12:00 NOON
12:05
12:10
12:15
12:20 P.M.
TIME

FIG. 19.

shortly afterwards, at about 12.19, a signal-check caused the speed to be reduced to about 10 miles per hour, as shown at C. The method of finding points on the curve of velocity is shown at B. The ordinate at B is proportional to the ratio $Aa : Ea$, where AE is a tangent to the displacement curve at A.

By the construction of such curves we can trace out the whole history of the motion, and if a sufficient number of points have been taken, and our drawing has been accurate, the results will be trustworthy for all practical purposes.

It is often difficult to draw the tangents to the curve correctly enough, especially if the slope of the curve is small, and it is usual to adopt another construction, to be explained later, which avoids the necessity of drawing the tangents by guesswork.

It is to be noted that diagrams of displacement may quite well be drawn, in which ordinates represent angles instead of distances, and from such diagrams angular velocities can be obtained exactly as described above.

19. Diagrams of Acceleration.—If a diagram of velocity on a time base be drawn, the curve of acceleration can be obtained from it by an application of the same method adopted for getting the velocity curve from that of displacement. In Fig. 20 let OAB be a curve of velocity. At any point A the rate of change of velocity is the limiting value of the ratio of the small change Δv in velocity to the small interval of time Δt in which such small change occurs, and by similar reasoning to that in § 18 it will be seen that this ratio is numerically equal to the tangent of the angle of slope of the curve at the point considered. In the figure, then, the acceleration at A is represented by tan α. Plainly at such a point as C, where the velocity has a maximum value (having ceased to increase and being about to decrease), there will be no acceleration, and the acceleration curve will pass through the time axis, as at c. Further, it will be noticed that where the velocity is increasing uni-

formly (as between D and E), and the curve of velocity is therefore straight, the acceleration curve becomes a horizontal straight line, as at *de*.

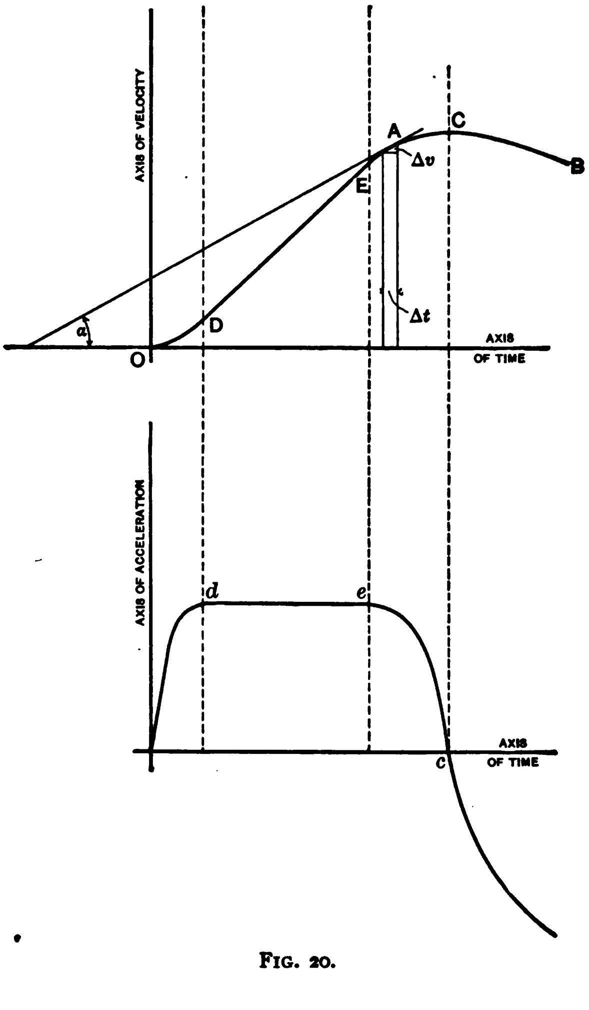


FIG. 20.

In practice, when such diagrams have to be drawn either for curves of velocity or of acceleration, a somewhat different method is adopted (as shown in Fig. 21), based on

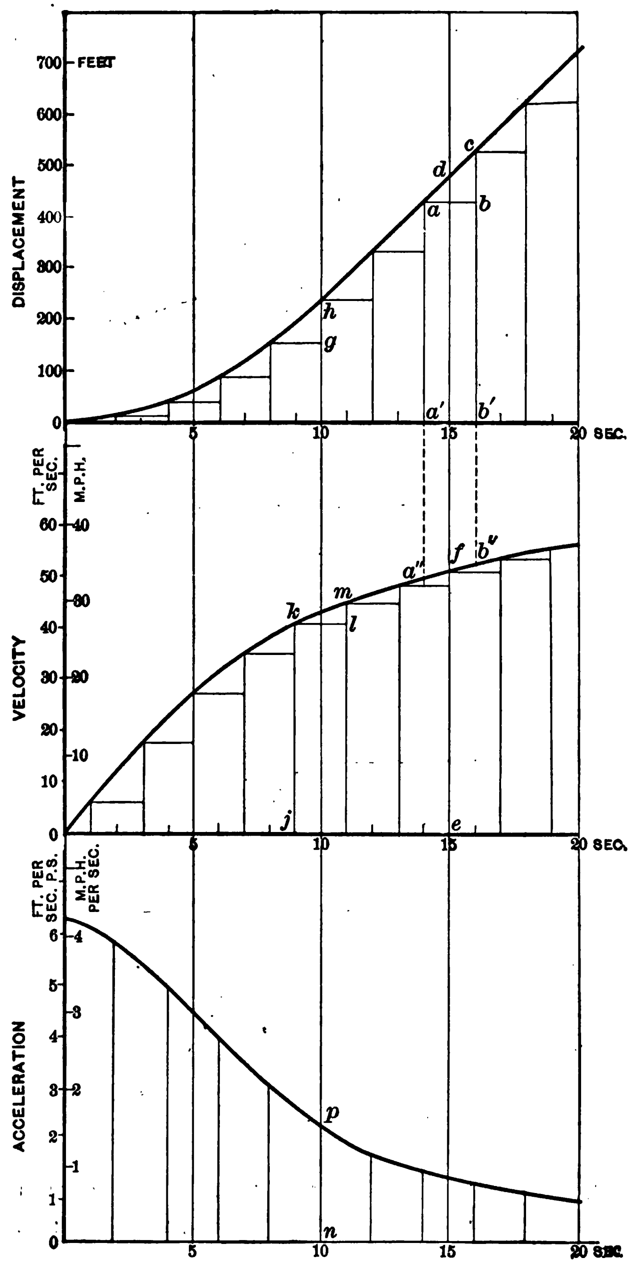


FIG. 21.

Distances from Rest.

5 sec.	66 ft.
10	245
15	490
20	750

Velocity.

5 sec.	19 m. p. h.
10	30
15	36
20	39

Acceleration.

5 sec.	3 m. p. h. per [sec.
10	1.4
15	.9
20	.6

the same principle, but avoiding the necessity of drawing a number of tangents to the curve, many of which can only be determined approximately.

The diagram shows a curve of displacement* for an electric street-car starting from rest. The data were obtained from tests of a special car designed for an initial acceleration of 3 miles per hour per second.

To construct from such a curve the corresponding curve of velocity, the time shown by the diagram is divided into a number of small intervals, in this case of 2 seconds each, as at *ab*. On measurement the length *bc* is found to represent 100 feet and is the difference between *aa'*, or *bb'*, and *cb'*. Now *aa'* is the distance moved by the car during the first 14 seconds. Thus *cb* represents the distance traversed during the fifteenth and sixteenth seconds, and accordingly the *average velocity* for those two seconds was $\frac{100}{2}$, or 50 feet per second.

In the figure an ordinate *ef* of length five times *bc* has been marked off at the point corresponding to 15 seconds from the start, and its extremity gives one point on our curve of velocity. In the same way *jk* has been made $5 \times gh$, and so on.

A curve drawn through the points thus found shows approximately what was the velocity at any time after the start. We say approximately, because the *actual velocity* at the middle of a 2-second interval would only be equal to the *average velocity* during such an interval if the straight line joining *ac* (for instance) had been parallel to the tangent to the displacement curve at *d*; that is, if the points *a''*, *f*, *b''* had been in a straight line. We know that actually this may or may not be the case, but by taking sufficiently small intervals of time we can reduce the error from this cause to any desired extent, until in fact it becomes negligible.

In order to obtain from the velocity curve that of accel-

* Taken from *Engineering News*, Oct. 14, 1897.

eration, an exactly similar procedure is employed. The length *lm*, for example, has been set up and exaggerated fivefold at *np*, and is proportional to the change of velocity during the tenth and eleventh seconds.

Having drawn our curves, it becomes necessary to determine their *scales*, that of the original diagram of displacement being known.

The displacement diagram was drawn originally * on such a scale that *bc* (representing 100 feet) was actually half an inch; the scale of distance was then 200 feet to the inch. The scale of time was $1'' = 5$ seconds, or 1 second $= \frac{1}{5}$ of an inch. A length *bc* transferred to the velocity diagram then represented (if $ab = 2$ seconds) a velocity of 50 feet per second, giving a scale of $\frac{1}{2}'' = 50$ feet per second, or 100 feet per second to the inch. For clearness this was exaggerated in the figure, so that 20 feet per second $= 1$ inch. In the same way the scale of the acceleration curve was made such that five inches $= 10$ feet per second per second. Scales of miles per hour and miles per hour per second have also been marked for comparison.

In general, then, the scales of such diagrams may be determined as follows:

Let the displacement diagram be drawn to a distance scale of 1 inch $= l$ feet, and suppose the short intervals of time during which the average velocities are estimated are each n seconds.

An ordinate of 1 inch in length on the displacement diagram when transferred to the velocity diagram then represents a velocity of l feet in n seconds, i.e., l/n feet per second. The scale of the velocity diagram is then n/l inches $= 1$ foot per second

Thus in the figure above we should have, if the ordinates were not exaggerated, a velocity scale of $\frac{1}{100}$ inch $= 1$ foot per second, for $l = 200$ and $n = 2$. It was drawn actually to a scale of $\frac{1}{20}$ inch $= 1$ foot per second.

* It is of course reproduced here to a smaller scale.

Considering next the scale of the acceleration diagram, suppose that on the velocity diagram the velocity scale is 1 inch $=m$ feet per second, while the small intervals of time are as before n seconds each; then an ordinate of 1 inch on the acceleration diagram will represent an acceleration of m/n feet per second per second.

In the figure $m=20$, $n=2$, so that the acceleration scale would naturally have been $\frac{20}{2}=10$ feet per second per second to the inch, had it not been exaggerated for clearness to 2 feet per second per second to the inch.

20. Diagrams on a Displacement Base.— Diagrams of velocity and acceleration may also be drawn *on a displacement base*, in which case lengths measured horizontally are proportional to distance traversed or angle described; such a construction is frequently very useful.

We have seen that on a time base the velocity curve for a body having uniform acceleration will be a straight line, passing through the origin of the two axes if the body has no velocity when time is reckoned zero. Velocity is then proportional to time. Suppose, however, that we consider the way in which velocity varies with regard to displacement in such a case.

Let a be the constant acceleration, v the velocity attained from rest after moving for a time t; then by our definition $a=\dfrac{v}{t}$. Now the distance moved by the body will be numerically equal to the average velocity $\left(\dfrac{v}{2}\right)$ multiplied by the time; thus $s=\dfrac{vt}{2}$ or $v=\dfrac{2s}{t}$.

But $v=at$; hence

$$v^2=\frac{2ast}{t}=2as.$$

Therefore, the acceleration being constant, the displacement varies as the square of the velocity. In the figure the

acceleration is 1.5 feet per second per second, and the distance traversed in the first two seconds (during which a

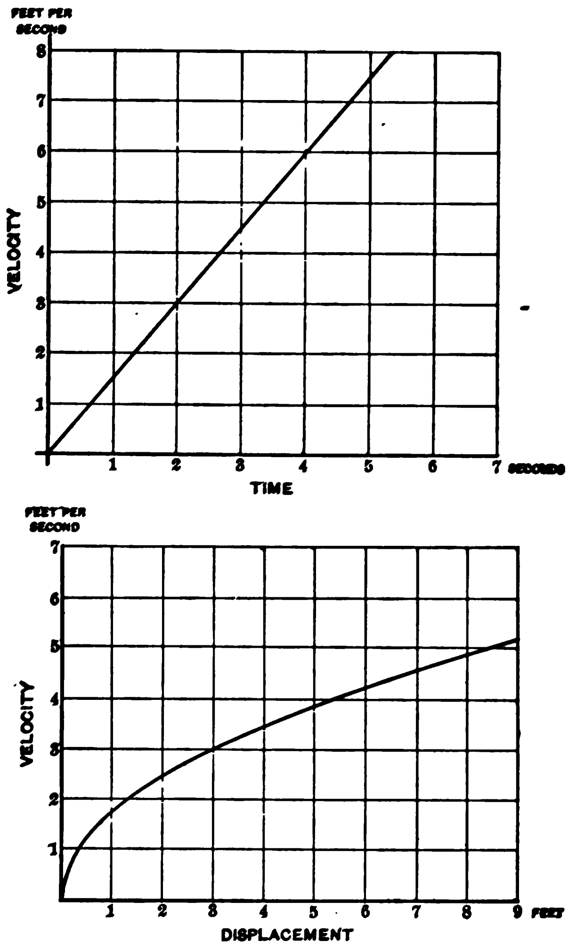


Fig. 22.

velocity of 3 feet per second has been attained) is $\frac{3}{2}\times 2 =$ 3 feet. In 3 seconds, the velocity being 4.5 feet per second, a distance of $\frac{4.5}{2}\times 3 = 6.75$ feet will have been described, and

in 1 second, the velocity being 1.5 feet per second, a distance of $\frac{1.5}{2} \times 1 = 0.75$ feet will have been covered. The diagram of velocity and displacement shown in the lower part of Fig. 22 will be found to express these relations, and the velocity curve on a displacement base is not a straight line, but a curve whose ordinates are proportional to the square roots of the abscissæ.

21. Acceleration Curves on a Displacement Base.—Let a, c, (Fig. 23), be two points on a velocity curve drawn on a displacement base. The difference of the ordinates bc represents the change of velocity. Draw the straight line ac, bisect

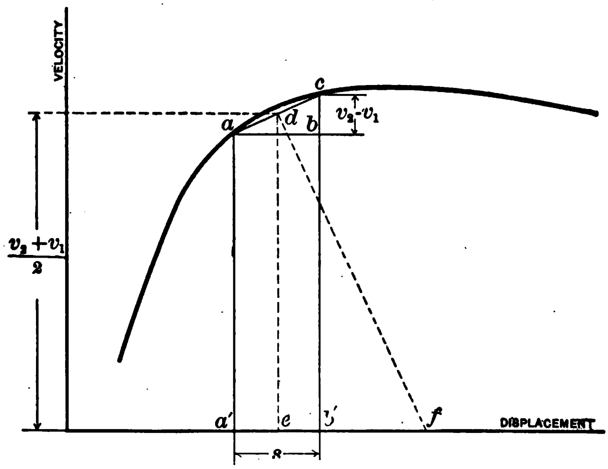


FIG. 23

it by the line df at right angles, and draw de perpendicular to the axis of displacement. Then $\frac{ef}{ed} = \frac{bc}{ab}$, since the triangles abc, def are similar.

Assuming that the velocity changes uniformly, while the moving body describes the distance represented by $a'b'$, let s be the space described in time t, while the velocity changes from v_1 to v_2. Then we know that

$$s = \frac{v_2 + v_1}{2}t \quad \text{and} \quad t = \frac{2s}{v_2 + v_1}.$$

Again, the acceleration

$$a=\frac{v_2-v_1}{t};$$

hence

$$a=\frac{(v_2-v_1\ (v_2+v_1)}{2s}.$$

But in the figure $aa'=v_1$, $cb'=v_2$, so that $v_2-v_1=bc$, $\frac{v_2+v_1}{2}=de$, and $s=ab$; thus

$$\text{acceleration}=\frac{bc\times de}{ab}=ef.$$

Actually, of course, the velocity in general does not change uniformly; we may, however, take a indefinitely close to c, so that the straight line ac becomes a tangent to the velocity curve, and df becomes the normal to that curve, while ef is the subnormal at d and represents the acceleration.

We find, then, that in the case of a velocity curve drawn on a displacement base the subnormal at any point represents the acceleration. It only remains to determine the scale on which, for example, ef represents the acceleration at e.

Let the scale for velocity be 1 inch $=m$ feet per second, while the distance scale is one inch $=n$ feet. Let ab, bc, de, and ef be measured in inches. Then, *numerically*, $v_2-v_1=bc\times m$ feet per second, $\frac{v_2+v_1}{2}=de\times m$ feet per second, and $s=ab\times n$ feet. But we have just seen that

$$\begin{aligned}\text{acceleration}&=\frac{(v_2-v_1)(v_1+v_2)}{s\times 2}\\&=\frac{bc\times m\times de\times m}{ab\times n}\\&=ef\times\frac{m^2}{n},\end{aligned}$$

ef being also measured in inches.

In Fig. 24 is given the curve of velocity for a cable-car from starting to stopping on a run of 200 feet, and from it the curve of acceleration has been drawn, the construction for one ordinate being shown. Note the high positive acceleration at the start, indicating a considerable jerk, followed (after the first 35 feet) by a small and variable value of the acceleration, sometimes positive, sometimes negative. The stoppage of the car is accomplished in the last 35 feet, and is shown by the negative acceleration or retardation during that portion of its travel.

As originally drawn, the velocity scale was 1 inch = 5 miles per hour = 7.33 feet per second, while the distance scale was one inch = 50 feet. Accordingly the acceleration scale was 1 inch $= \frac{7.33 \times 7.33}{50} = 1.075$ feet per second per second, or 0.732 mile per hour per second. The greatest acceleration is then about 3.3 miles per hour per second, and the greatest retardation about 2.2 miles per hour per second.

It should be noted that the *curve of velocity on a displacement base, when acceleration is constant, is a parabola*, this being the curve one of whose characteristic properties is the constancy of the subnormal.*

22. Polar Diagrams of Displacement, Velocity, and Acceleration.—Besides employing the methods just given for drawing curves of displacement, etc., it is often useful, especially in considering periodic motion (in which the same circumstances or conditions as to velocity or displacement repeat themselves at regular intervals of time), to draw diagrams in which a radius vector represents displacement, velocity, or acceleration, while the angle turned through by such a radius is proportional to time.

In Fig. 25 suppose that a line ON turns about the point O, starting from the initial position OM; the angle θ it has described, when it has reached any position such as OA, being proportional to an interval of time t_1, during which a body

* See Fig. 22.

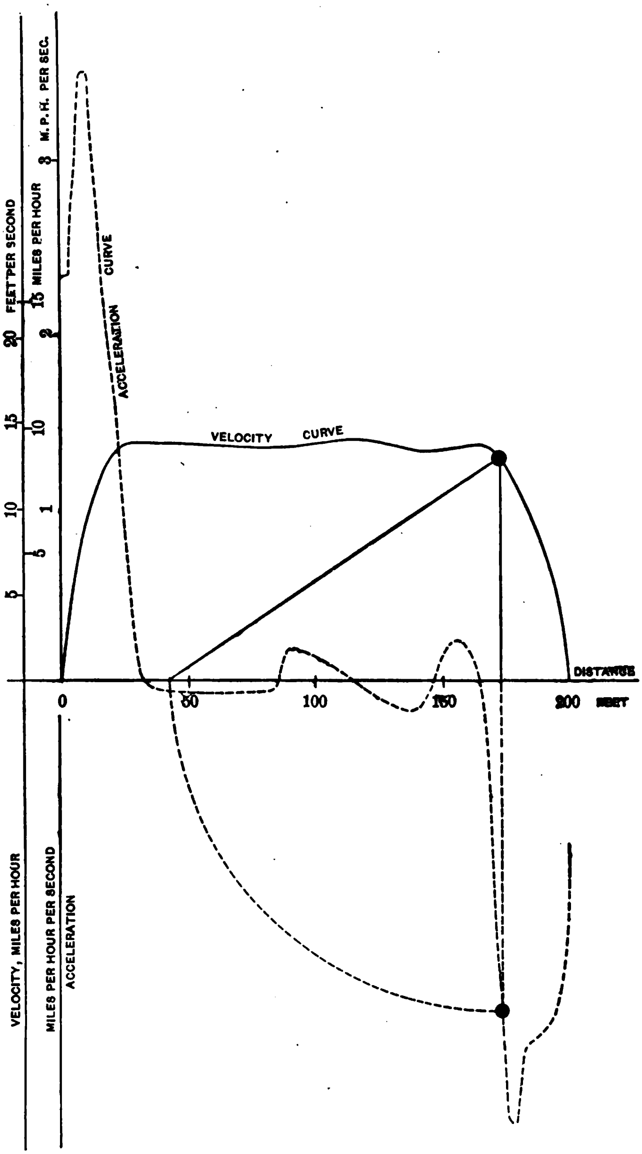

FEET PER SECOND
MILES PER HOUR
M.P.H. PER SEC.
ACCELERATION CURVE
VELOCITY CURVE
DISTANCE
FEET
VELOCITY, MILES PER HOUR
MILES PER HOUR PER SECOND
ACCELERATION
0
50
100
150
200

FIG. 24.

whose motion we are considering has moved to a distance s_1 from its starting-point. Mark off a distance $OA = s_1$. Then a curve, such as NAB, drawn through successive positions

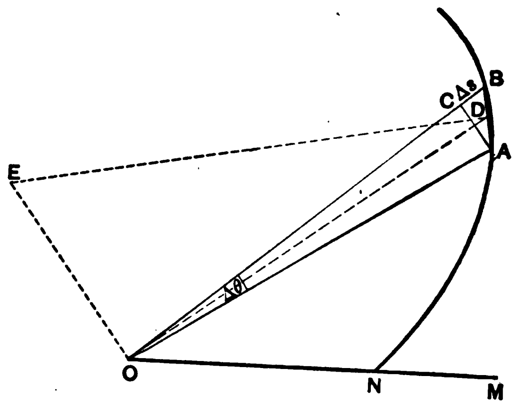


FIG. 25.

of A, is a *polar diagram of displacement* for the body considered. Let $OB = s_2$, while the angle $BOM = t_2$.

Then the average velocity between the times t_1 and t_2 will be represented by $\dfrac{BO-AO}{\text{angle } BOA} = \dfrac{s_2-s_1}{t_2-t_1}$. Making $OC = OA$, we have, therefore,

$$\text{average velocity} = \frac{BC}{\text{angle } BOA} = \frac{\Delta s}{\Delta \theta}.$$

Join AC, and draw the straight line BA. Now if $\Delta\theta$ is a *very small angle*, the length of the straight line AC does not differ sensibly from that of the arc of a circle of radius OA, so that we may say that if $\Delta\theta$ is small,

$$\Delta\theta = \frac{AC}{AO};$$

$$\frac{\Delta s}{\Delta\theta} = \frac{BC.AO}{AC}.$$

Bisect AB in D, and draw OD, DE, OE, respectively perpendicular to AC, AB, BC, so that the triangles BCA, EOD are similar; then $\frac{BC}{AC}=\frac{EO}{DO}$ and

$$\frac{\Delta s}{\Delta \theta}=\frac{OE}{DO}\times AO.$$

Next suppose that the interval of time represented by $\Delta\theta$ becomes indefinitely small, so that the points B and A coincide, and AB becomes a tangent to the curve. Δs and $\Delta\theta$ will both become infinitesimal in magnitude, but their ratio (which now represents the velocity the body has when a time represented by θ has elapsed) is finite, and its value is measured by the limiting value of $\frac{OE.AO}{DO}$ when $AO=DO$.

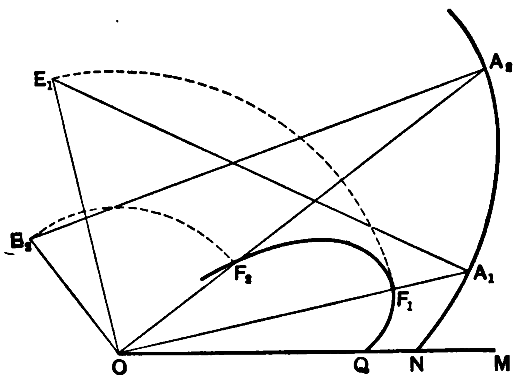


FIG. 26.

This is of course OE. Hence the velocity is measured by the length of OE when taken to the proper scale and drawn perpendicular to OA, and to find the velocity corresponding to any point A on a polar displacement diagram, we draw a normal to the curve, and find its intercept OE on a line drawn perpendicular to the radius vector OA. By carrying out this construction for a number of points and marking off values of OE along OA (as at OF, Fig. 26) in

each case, we find points on a new curve, QF_1F_2, which is in fact a polar velocity-time diagram.

The reader will now be in a position to see that by repeating the same construction with the new velocity curve a further diagram is obtained, that of acceleration; since acceleration has exactly the same relation to velocity that velocity has to displacement. In this case, of course, the quantity represented by a line drawn in the same way as OE above is the rate of change of *velocity* with regard to time.

We may next study some examples of such diagrams.

Fig. 27 represents a polar diagram or curve of displacement, in which distance increases uniformly with time, as shown by the fact that $QO - PO = PO - NO$, if the angle QOP = angle PON, and so on. The curve $ONPQ$ is of course an Archimedean spiral. It may be proved that the length of the intercept OE by the normal DE on a line OE through O perpendicular to the radius vector is constant whatever the position of D* on the curve. This length OE has been shown to represent the velocity, and accordingly in this case the velocity diagram will be a circle with radius OE, and the acceleration will be zero, since the velocity is uniform. In the drawing, if the displacement scale is as shown, and the time scale is $90° = 10$ seconds, the velocity of the body is 1 foot per second. To determine the velocity scale of such a diagram we need only reflect that

* The polar equation to the spiral of Archimedes is $r = a\theta$;

$$\text{hence } \theta = \frac{r}{a} \text{ and } \frac{d\theta}{dr} = \frac{1}{a}.$$

If ϕ be the angle which the tangent makes with the radius vector,

$$\tan \phi = r\frac{d\theta}{dr}.$$

Thus $\tan \phi = \dfrac{r}{a}$.

Now the angle between the normal DE and a line OE perpendicular to the radius vector is ϕ. Hence $OE = \dfrac{r}{\tan \phi} = r.\dfrac{a}{r} = a =$ constant.

the velocity is numerically $\frac{ds}{d\theta}$. If the displacement scale were 1 inch = 3 feet, a velocity of 1 foot per second would be represented by a length of one third of an inch divided by the angle representing 1 second. In this case

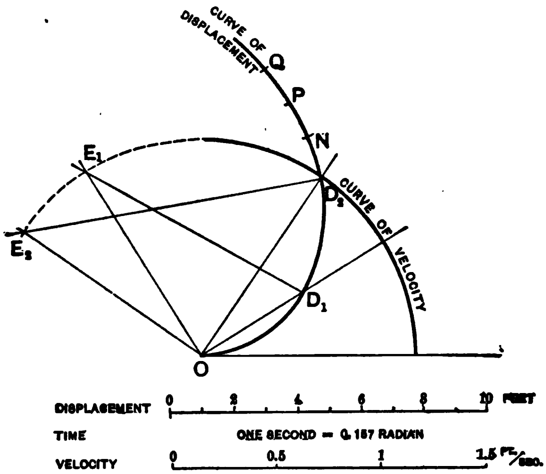


FIG. 27.

90° represents 10 seconds of time, so that 1 second of time would be denoted by $\frac{\pi}{20}$ or 0.157 radian. Hence unit velocity would be represented by $\frac{1}{3 \times 0.157} = 2.12$ inches, and the velocity scale would be 1 inch = 0.471 foot per second.

23. **Diagrams for Simple Harmonic Motion.** — In the following chapter we shall frequently meet with cases in which bodies have periodic motion in a straight line, either exactly or approximately harmonic in character. We define *simple harmonic motion* as the motion of a point which is the orthogonal projection on a straight line of another point moving uniformly in a circle, termed the *auxiliary circle*. The radius of the auxiliary circle is called the *amplitude* of the motion. Such motion can be conveniently studied by means of polar diagrams; the engineer, for example, often employs such diagrams to elucidate the action of the slide-valve of a steam-engine, for the valve

has very nearly such a motion as has been defined above. In Fig. 28 let AOB represent the path of a point having

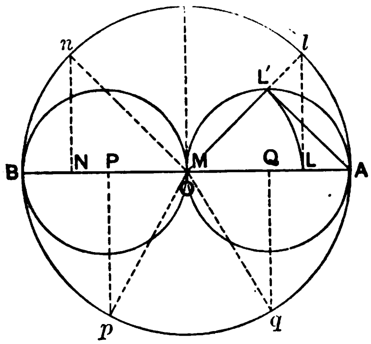


FIG. 28.

simple harmonic motion; we wish to draw diagrams of displacement, velocity, and acceleration for this point. AO is the amplitude of the motion, $AmBq$ is the auxiliary circle, and, according to our definition, $LMNP$, etc., will be positions of the point after the lapse of times proportional to the angles AOl, AOm, AOn, AOp, and so on. It is plain, then, that a polar time-displacement diagram can be readily drawn by marking off along each radius a distance equal to the corresponding displacement of the vibrating point from its mid-position. Thus OL' has been made equal to OL. It should be noticed that during each revolution of the rotating point round the auxiliary circle, its projection travels from A to B and back again, so that twice during each period the displacement of the vibrating point will be zero, while at the same time the velocity will have its greatest value.

The *period* is of course the time of a complete revolution of the point round the auxiliary circle. During one half of this time the vibrating point has a positive velocity, i.e., is moving in one sense, say from right to left, while during the remainder of the period the velocity will have a negative value.

The displacement diagram is drawn by obtaining a series of such points as L', where $OL' = OL$; the locus of such points is easily shown to be a pair of circles touching the auxiliary circle and each other. Taking the point L', for example, join $L'A$. Then in the triangles OlL, OAL' we have $OL' = OL$ and $Ol = OA$, while the angle lOA is common to both. Hence the triangles are equal in all respects, and the angle $OL'A$ is a right angle. Therefore the point L' lies on a semicircle drawn on OA as a diameter, and the complete locus of L' is a pair of circles, as shown. The radius vector OL' represents the displacement of the vibrating point at a time represented by the angle between the radius vector and the initial line OA.

Having given such a diagram of displacement (Fig. 29), let us apply to it our construction for determining velocity. At any point D the line DE is drawn normal to the curve of displacement, and is cut by OE where the angle EOD is a

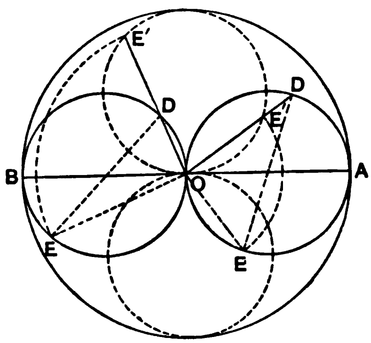


FIG. 29.

right angle. Accordingly, when the length OE is marked off at E' along OD we get one point on the velocity diagram. Plainly, in this particular case, any radius vector of the velocity diagram, such as OE', is equal in length to the radius vector of the displacement diagram which makes an angle of 90° with it, and it follows that the velocity diagram will

also be a pair of circles, as shown in the figure. Their axis is, however, at right angles to the axis of the displacement diagram, the maximum velocity being reached when the body is at the middle point of its path.

On constructing the diagram of acceleration, we find that it also takes the form of a pair of circles and coincides with that of displacement. The scales of the two diagrams are of course not the same, but it follows that *in simple harmonic motion the acceleration at any instant is proportional to the distance of the vibrating point from its mid-position*, a fact which can also be readily proved analytically.*

Fig. 30 gives the linear and polar diagrams of displacement, velocity, and acceleration for a simple harmonic motion of which the amplitude is 0.75 foot, while the period is $\frac{1}{5}$ second. This corresponds approximately, but by no means exactly, to the motion of the piston of a steam-engine, 18 inches stroke, and making 300 revolutions per minute.

The linear diagrams have been plotted on a straight line base from the radial diagrams; they might have been drawn by the methods of §§ 18 and 19, in which case their scales would have been different from those shown. In general, polar diagrams are easier to draw than linear diagrams, and are more useful for cases of periodic motion such as this; the linear diagrams are added for the sake of comparison.

With regard to the scales of these figures, the scale of time as originally drawn was $180° = 0.1$ second (i.e., 1 second $= \frac{2\pi}{0.2} = 31.416$ radians), while that of distance was 1 foot $= 1$ inch. Accordingly unit velocity was represented by a length of $\frac{1}{31.416}$ inch. The velocity scale was thus 1 inch $= 31.416$ feet per second. Unit acceleration was represented by

* See Perry, Applied Mechanics, p. 549.

$\frac{1 \text{ foot per second}}{1 \text{ second}}$, or $\frac{.0318}{31.416} = .001011$ inch, hence the acceleration scale was 1 inch = 988 feet per second per second.

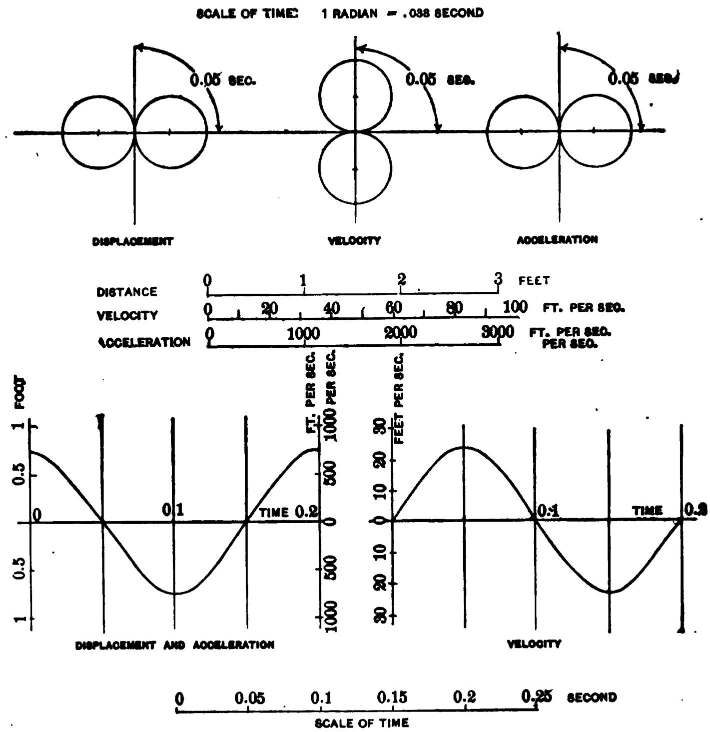


FIG. 30.

Note that the velocity of the rotating point round the auxiliary circle is $\frac{1.5\pi}{0.2} = 23.6$ feet per second. The maximum velocity of the vibrating point as measured from the diagram is $0.75 \times 31.416 = 23.6$ feet per second, thus agreeing with our definition of simple harmonic motion. The maximum acceleration of the vibrating point will be found to be the same as the radial acceleration of the point

travelling round the auxiliary circle, namely $\frac{v^2}{r}$, or $\frac{23.6 \times 23.6}{0.75} = 741$ feet per second per second.

If an acceleration diagram on a displacement base is required, this can easily be drawn from the acceleration-time and displacement-time diagrams, and it will be found to take the form of a straight line, since, as has previously been remarked, acceleration in simple harmonic motion is proportional to displacement.

24. Relative Motion of Two Bodies each having S. H. M.—Cases arise in which it is necessary to find the relative

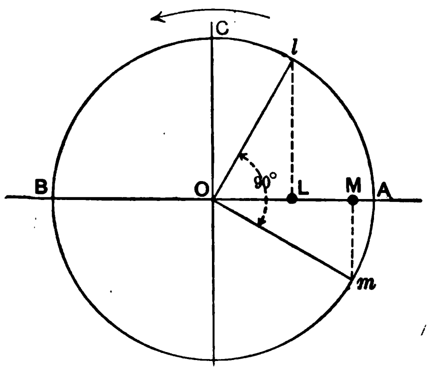


FIG. 31.

motion of two or more bodies having simple harmonic motion, or we may wish to determine the motion resulting from the combination of two or more simple harmonic motions. We proceed to show how this is done.

It has been seen that every simple harmonic motion is capable of being referred to a corresponding uniform motion round a circle called the auxiliary circle. At any instant, for example, the simple harmonic motion of the point L (Fig. 31) corresponds to the uniform motion round the circle of the point l.

Let there be two points, L and M, having simple har-

monic motion (of the same or different amplitude), and let l and m be their reference points. If AOB is the line on which the motion of l and m is projected, we define the constant difference between the angles COl and COm as the *difference of phase* in the two simple harmonic motions. In Fig. 31 the point M has a motion differing in phase by $-90°$ from that of the point L; in other words, Om continually lags $90°$ behind Ol, and of course M lags behind L to a corresponding extent.

In Fig. 32 let there be two points, L and M, having simple harmonic motions along AOB, whose amplitudes are

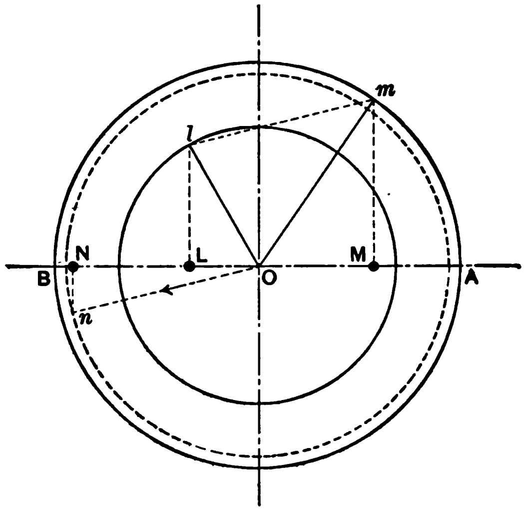


FIG. 32.

Ol and Om, and whose difference of phase is the angle lOm, the period being the same for both motions. We wish to find the relative motion of L and M, compounded of their two simple harmonic motions.

Join lm and draw On equal and parallel to ml. Draw nN perpendicular to AB.

Then the relative displacement of L and M is $OM-OL$,

where OM or OL is reckoned negative when measured to the left of O. Now $OM - OL = ML$ is the projection on AB of the line ml. Also, ON is the projection on AB of the line On, equal and parallel to ml. Thus $ON = ML$, since each is equal to $On \cos NOn$.

Hence in any position the relative displacement of M and L is equal to the distance of the point N from the centre O. But N has a simple harmonic motion of the same period as that of M and L, of amplitude $On = ml$, and differing in phase from that of L by the angle lOn. Thus if two points have simple harmonic motions of the same periods and along the same straight line, their relative motion is also a simple harmonic motion, in general differing in amplitude from either of the components, and also differing in phase.

In Fig. 33 diagrams of displacement are drawn for two simple harmonic motions of 2 seconds period, the amplitudes being $\frac{3}{4}$ inch and 1 inch, and the phase difference 60°.

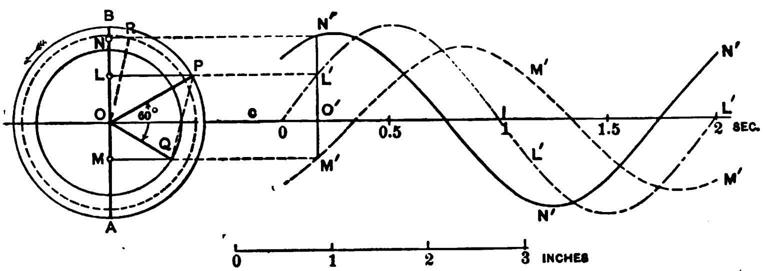


FIG. 33.

The curves L' and M' have ordinates proportional to the displacements of the corresponding points along AB, as represented by the motion of the points PQ round their respective auxiliary circles. The resultant displacement of L relatively to M is shown by the distance $M'L' = O'N'$, N' being above or below the line CD (along which time is measured) according as L is above or below M along the line AB.

Thus the curve N' is drawn through points whose heights above CD are equal to the heights of L above M at different times. It will be found that these displacements are the same as those of a point N moving along AB with a simple harmonic motion of period 2 seconds, amplitude 0.9 inch, whose phase is about 45° behind that of L and 105° behind that of M.

The curves drawn in Fig. 33 are in fact sine curves, since their ordinates are proportional to the sines of angles which are proportional to the abscissæ. In general, on compounding simple harmonic motions of different periods we do not obtain a simple harmonic motion as a result, but a more complex movement which is still, however, *periodic*. A case of this is shown in Fig. 34, where two simple har-

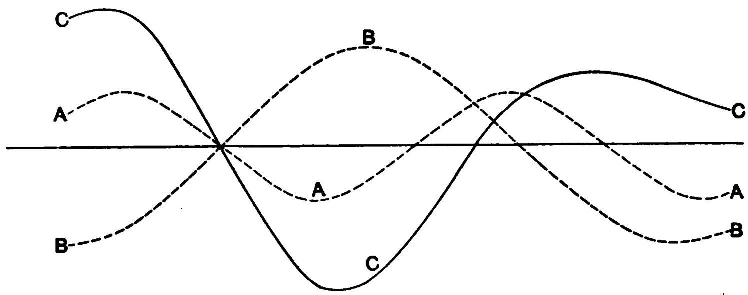


FIG. 34.

monic curves, A and B (shown by dotted lines), differing in phase, amplitude, and period, are compounded, the resultant curve C being shown by a full line. It is possible to resolve the curve representing any periodic function into a number of component sine curves of different phase, amplitude, and period. In this way, for example, the complex curve drawn by a tide-gauge is analyzed, and the periods, amplitudes, and phases of its component tides are determined.

In the figure, for instance, the curve CC might represent

the actual rise and fall of the water-level at a certain place, due to the simultaneous effects of two tides, which, acting alone, would respectively produce the fluctuations shown by AA and BB. Distances measured horizontally represent time as before.

Such diagrams as those given above enable us to study any periodic function, and they find many important applications in scientific work.

25. Composition of S. H. M. not along Same Line.—Imagine that a point, having simple harmonic motion in the direction of a given line AB, has impressed upon it another simple harmonic motion in the direction of a line CD at right angles to AB. The motion of the bob of a simple pendulum is very approximately simple harmonic. An arrangement like that shown in Fig. 35 may be devised, which will give to a pencil, P, a motion compounded of those of two pendulums, Q and R, swinging in planes at right angles to one another.

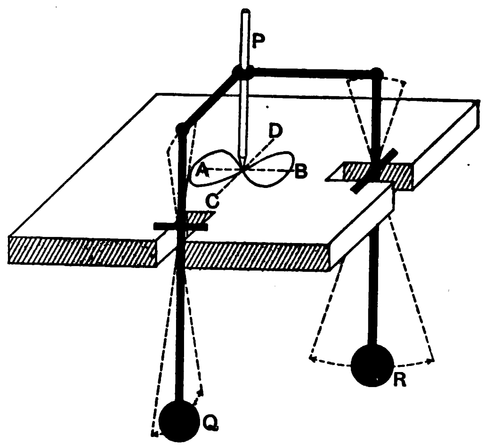
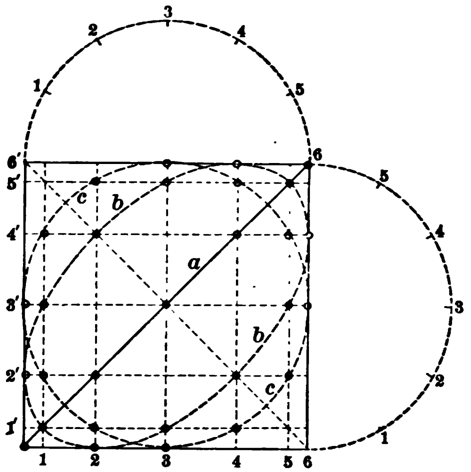


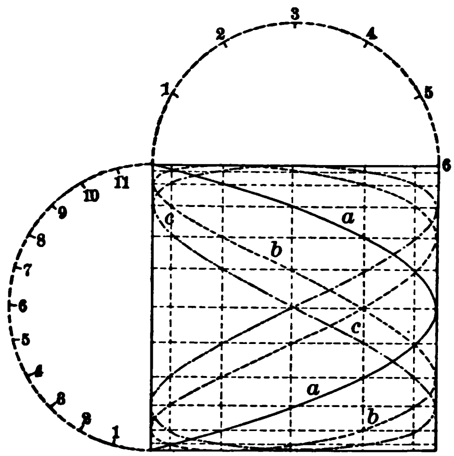
FIG. 35.

It is easy to see what the path of such a pencil will be. In Fig. 36 let 0 1 2 3 4 5 6 represent successive positions of the



PERIODS EQUAL, AMPLITUDES EQUAL
a, SAME PHASE
b, PHASE DIFFERENCE 60°
c, " " 90°

FIG. 36.



PERIODS 2:1, AMPLITUDES EQUAL
a, INITIAL PHASE DIFFERENCE 0°
b, " " " 60°
c, " " " 90°

FIG. 37.

vibrating point measured along one line, and o 1′2′3′4′5′6′ those measured along the other, equal intervals of time being taken, and the periods of the two motions being equal.

In the figure the movement from 0 to 1 or 1 to 2 is executed in $\frac{1}{12}$ of a complete period. Now if both harmonic motions have the same phase, points of intersection of the lines drawn through 1 and 1′, 2 and 2′, 3 and 3′, etc., parallel to the axes OA, OC, will give successive positions of the tracing-point. If there is a phase difference of 60°, the tracing-point will have moved as far as 2′ along one line, while it

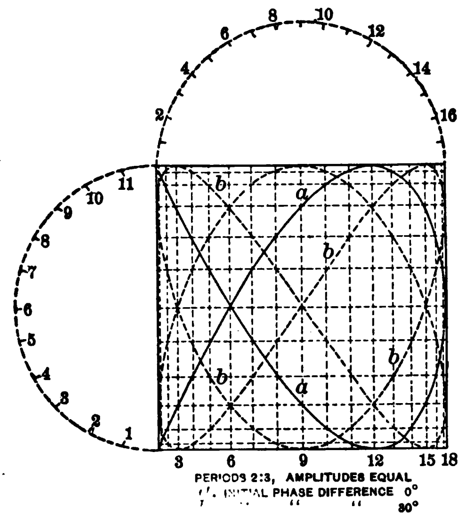


FIG. 38.

is still at 0 on the other; hence the lines drawn through 1 and 3′, 2 and 4′, and so on, now give the path, which is seen to be an ellipse. Again, with a phase difference of 90° the path becomes circular. Figs. 37 and 38 show the curves resulting from the compounding of two simple harmonic motions of equal amplitudes, having periods in the proportion of 2 : 1 and 2 : 3 respectively, and various initial phase differences The combination of motions which have periods in any other ratio can readily be illustrated by the same method, and the reader will find it instructive to plot for himself some of the resulting curves.

CHAPTER III.

PLANE MECHANISMS CONTAINING ONLY TURNING PAIRS.

26. Quadric Crank-chains.—If we endeavor to make a plane mechanism out of links containing only turning pairs, we find that the least number of links with which this can be done is four. A chain of *three* links so connected forms

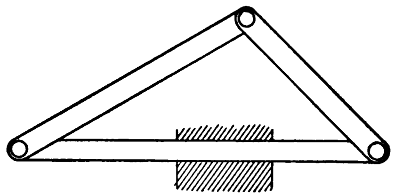


FIG. 39.

an arrangement which is of value as a *structure* (a simple triangular roof-truss), but is of no service as a mechanism, since its parts can have no relative motion.

On the other hand, a simple chain of *five* or any greater number of links connected by turning pairs is equally useless as a mechanism, since the relative motion of at least two of its links is not constrained, as has been shown in § 3.

Let us consider, therefore, a chain of four links connected by turning pairs whose axes are parallel. When the links of this chain are of unequal lengths the smallest is called the crank, and since the four links form a quadrilateral, the chain has been called by Reuleaux * the *quadric (cylindric) crank-chain.* The term 'cylindric' distinguishes this chain

* Kinematics of Mach., §§ 62–65.

from the corresponding spheric chain, in which the axes are not parallel.

In quadric crank-chains it will be convenient to distinguish between links having a swinging or partial turning movement and those which can execute complete rotations relatively to the fixed link in the chain.

The former links will be called levers, the latter cranks. It is obvious that by altering the relative lengths of the links we can obtain different relative motions, and hence different mechanisms. From these, again, other different mechanisms are produced by inversion of the chain.

27. **Virtual Centres and Centrodes.**—Let *abcd*, Fig. 40, represent the four links of a quadric crank-chain. Each of these links will have motion relatively to every other, and hence we shall have six virtual centres. Four of these centres are readily identified as the axes of the turning pairs; for instance, the virtual centre of *c* with regard to *d*, or of *d* with regard to *c*, is obviously the point 3, and may be

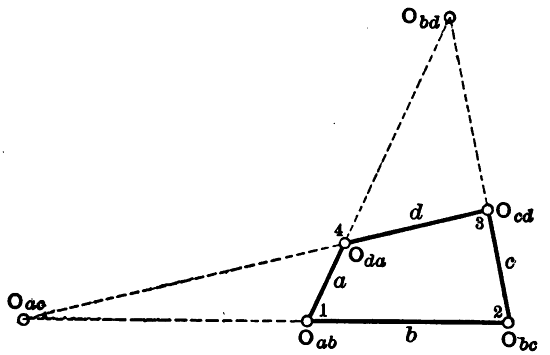


FIG. 40.

indicated as O_{cd} or O_{dc}. In the same way we have O_{ad}, O_{ab}, and O_{ac}; all these points are in fact permanent centres as regards their own pair of links. Remembering that for any three bodies having plane motion the three virtual centres lie in one straight line, it is easy to see that O_{ac} must lie at

the join of the straight lines drawn through O_{ab} and O_{bc}, and through O_{ad} and O_{cd}. In the same way O_{bd} is at the intersection of the lines $O_{ab}O_{ad}$ and $O_{cb}O_{cd}$.

Supposing b to form the *frame* or fixed link, it is seen that since O_{ab} and O_{bc} are permanent centres, the centrodes of a and c with regard to b are points, namely O_{ab} and O_{bc}.

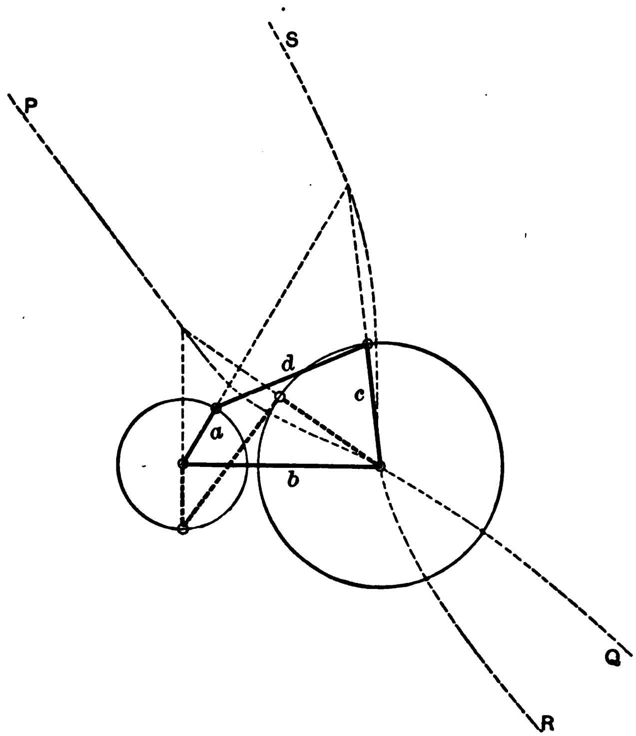


FIG. 41.

The centrode of d with regard to b is the locus of O_{db} and takes the form of a curve having four infinitely distant points; portions of it are readily drawn by finding a series of positions of O_{db} corresponding to successive positions taken up by the three links a, c, and d. In a similar way may be obtained the centrode of b with regard to d (supposing d to be the fixed link). The curve $PQRS$

in Fig. 41 represents the centrode of *d* with regard to *b*; the construction for two points on the curve is shown.

28. Angular Velocities.—It is frequently of importance, having given the angular velocity, say, of the link *a*, to find

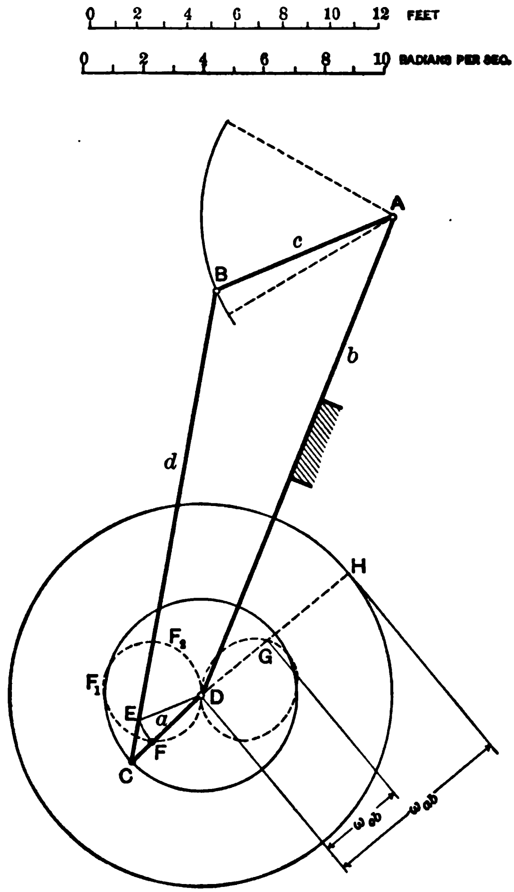


FIG. 42.

that of any other link, say *c*, or, in other words, to determine the angular velocity ratio of the chain. This can be very simply done by construction.

In Fig. 42 let *ABCD* represent the mechanism, *AD* being

the fixed link, and the uniform angular velocity of CD being known. It is required to determine the angular velocity of AB for any position of the mechanism.

Draw DE parallel to AB, and cutting BC, or BC produced, in E. With centre D and radius DE mark off DF

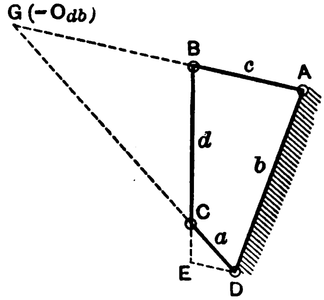


FIG. 43.

along DC. Then DF represents the angular velocity of c on the same scale as that on which AB represents the angular velocity of a, and if a series of points such as F be obtained, the curve FF_1F_2G . . . drawn through them will form a polar diagram of angular velocities for c and a.

To prove this construction, let ω_{cb}, ω_{ab} be the angular velocities of c and a respectively with regard to b. In Fig. 43 find O_{db}, the intersection of AB and DC at G, and draw DE parallel to AB, meeting BC in E.

Since the link d is turning for the instant about G, we must have

$$\frac{\text{linear velocity of } B}{\text{linear velocity of } C}=\frac{GB}{GC}.$$

Now $$\omega_{cb}=\frac{\text{linear velocity of } B}{AB},$$

and $$\omega_{ab}=\frac{\text{linear velocity of } C}{CD};$$

hence $$\frac{\omega_{cb}}{\omega_{ab}}=\frac{\text{linear velocity of } B}{\text{linear velocity of } C}\cdot\frac{CD}{AB}$$

$$=\frac{CD.GB}{AB.GC}.$$

But by construction the triangle BGC is similar to the triangle EDC; hence

$$\frac{GB}{GC}=\frac{ED}{DC}$$

Therefore $$\frac{\omega_{cb}}{\omega_{ab}}=\frac{CD.ED}{AB.DC}=\frac{ED}{AB}$$

Thus if AB represents the angular velocity of a with regard to b, ED represents on the same scale that of c with regard to b.

Fig. 42 gives such a velocity diagram, drawn to scale, for the beam of a beam-engine when the crank rotates uniformly. For comparison the circle of radius $DH=AB$ has been drawn, so that for any radius DGH the intercept DG represents ω_{cb}, just as DH represents ω_{ab}. The polar curve of velocity is shown by a dotted line.

The distances taken are:

$$AB=8 \text{ feet}=DH;$$
$$BC=20 \text{ feet};$$
$$CD=4 \text{ feet};$$
$$DA=21.5 \text{ feet}.$$

When the crank is in the position DH the angular velocity ratio is

$$\frac{DG}{DH}=\frac{3.5}{8}=0.438,$$

or at that particular instant the beam is swinging with 0.438 the angular velocity of the crank. If the crank rotates

uniformly at 60 revolutions per minute or 6.28 radians per second, in the position AB the beam is moving with an angular velocity of $6.28 \times 0.438 = 2.75$ radians per second.

From the curve of angular velocity thus obtained we might draw the curve of angular acceleration by the construction described in § 22. Notice that the construction just described can still be applied in positions of the mechanism where O_{bd} is inaccessible, i.e., when AB and CD are nearly parallel, and when the relative angular velocities, therefore, could not be found from the position of the virtual centres.

29. Inversions of the Quadric Crank-chain.—In the particular example of the quadric crank-chain just examined, the lengths of the links are such that while the link a executes complete rotations with reference to b or d, c only swings. a is then a crank, c a lever, and if the link b is the fixed one, the resulting mechanism is called the lever crank-chain.

In order that a may execute complete rotations with regard to b it is necessary that $a+b \leqq c+d$, while also $a+d \leqq c+b$, a being the smallest of the links.

With these proportions let us see the result of inversion of the chain. On considering the relative motions of the links we find that the motion of a relatively to b or d is that of complete rotation, while with regard to either of the

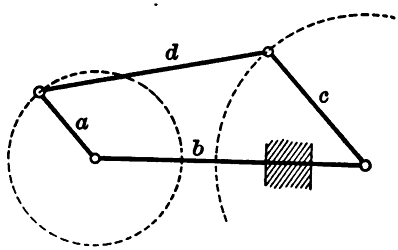


FIG. 44.

same links c only swings or performs partial revolutions. As has been already pointed out, inversion can make no

change in the *relative motions* of the links, and hence the mechanism will remain a *lever crank-chain* whether *b* or *d* be the frame (or fixed link).

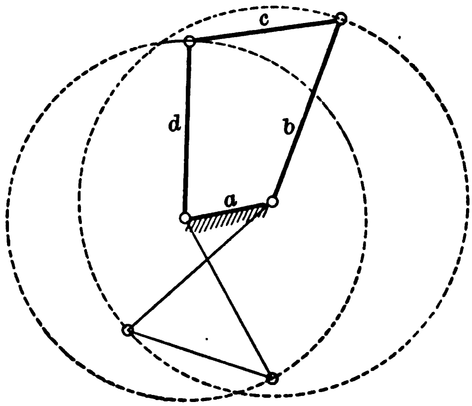


FIG. 45.

On fixing *a*, however, as in Fig. 45, a new mechanism is obtained which may be called the *double crank*, inasmuch as both *b* and *d* can now execute complete rotations about the axes of the pairs *ba* and *da*.

This mechanism is used in practice as a drag-link coupling, *b* and *d* being represented by the discs keyed on to the two shafts, *a* by the frame containing the bearings, and *c* by the drag-link connecting the pin on *d* with that on *b*. The mechanism is also employed in the construction of feathering paddle-wheels.

When used for this purpose the object is to cause the floats to enter and leave the water edgewise, (so as to avoid splashing,) while remaining vertical at the bottom of their travel.

Thus suppose *A*, *B*, *C*, Fig. 46, to be points on the path of the outer edge of a float, *AC* being the water-line. The steamer has a certain speed relatively to the water, so that a point *A* on the wheel is moving horizontally with a velocity

AD in common with every other point on the vessel, the length of *AD* thus representing the speed of the ship to any convenient scale. In virtue of the rotation of the wheel, *A* has also a linear velocity (represented by *AE* to the same scale), relatively to the ship; therefore the real direction in which

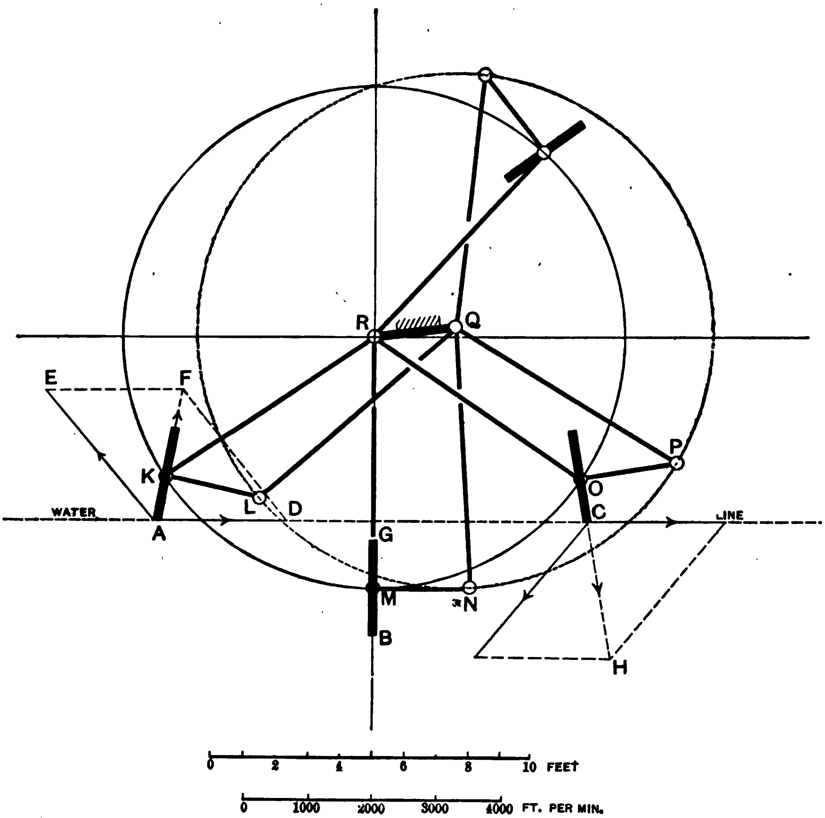


FIG. 46.

A moves relatively to the water is *AF*, the diagonal of the parallelogram *AEFD*. The floats then at entrance and exit should lie in the positions *AF* and *CH*, while the float in its lowest position should evidently be vertical, as at *BG*. The floats are pivoted at their centres to the framework of

the wheel and have attached float-levers KL, MN, OP. The ends L, N, and P are all connected by radius-rods to an eccentric-pin, generally fixed on the sponson-beam of the paddle-box. The centre of this pin is of course at Q, the centre of the circle passing through N, P, and L, while R is the centre of the wheel itself. It will be seen that the paddle-wheel arm, the float-lever, the radius-rod, and part of the ship's structure form a double-crank mechanism, thus giving the floats the desired movement. Fig. 46 is drawn to scale from the following data:

Speed of ship 21 knots = 2026 feet per minute.
Diameter of wheel (ext.) 18.5 feet.
Revolutions per minute 48.
Breadth of floats 3′ 0″.
Immersion of lower edge 3′ 6″.
Length of float-levers 3′ 0″.
Speed of outer edge of float 2790 feet per minute.

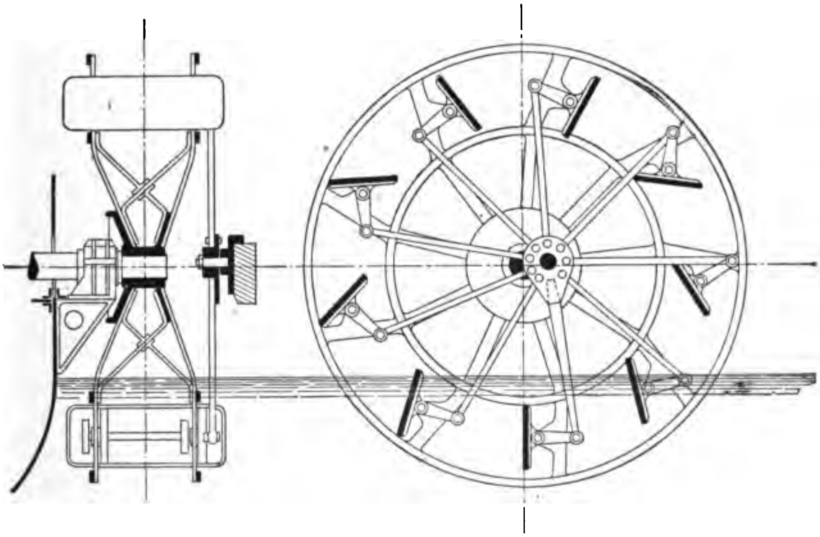


FIG. 46*a*.

Fig. 46*a* is a drawing of the arrangement of an actual feathering paddle-wheel. The reader will have

no difficulty in recognizing various links in the double-crank mechanism.

Next suppose c (Fig. 44) to be the fixed link. Remembering that the relative motions of b and c and d and c are partial and not complete rotations, we see that a third mechanism, the *double lever*, is the result of this inversion.

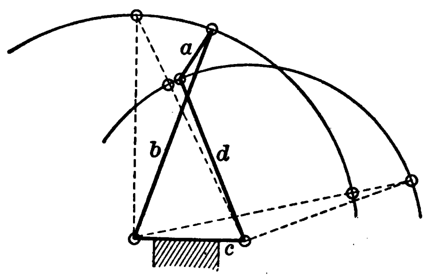


FIG. 47.

The double-lever mechanism is shown in Fig. 47, and such an arrangement finds an application in certain approximate straight-line motions (compare Fig. 56*b*).

The result of the various inversions of the quadric crank-chain may be summarized as follows:

Fixed Link.	Mechanism.
a	Double crank.
b	Lever-crank.
c	Double lever.
d	Lever-crank.

The four inversions have thus given us three different mechanisms.

Certain special cases of the quadric crank-chain have peculiarities which are of interest. Suppose that the lengths of the links are such that either $a+b=c+d$ or $a+d=c+b$, b being the fixed link. This condition is shown in Fig. 48.

As we have seen, it is still possible for a to execute complete rotations, but it will now be found that c can swing

on either side of the fixed link *b*. We obtain one position of the chain in which all the links are in a line, and the angle through which *c* can swing is then doubled. This condition is expressed by saying that the mechanism passes through

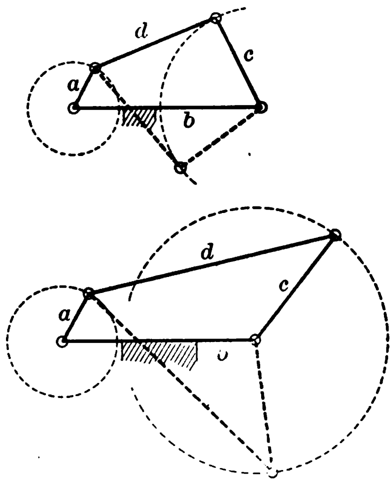


FIG. 48.

a *change-point* when all the links are in line, and for any given position of the link *a* the mechanism can assume either the position shown by the full lines or that shown by the dotted lines in the figure.

30. Change-points and Dead-points. — A *change-point* may be defined as a position of a mechanism in which such a want of constrainment occurs that it is possible for the arrangement to transform itself into another mechanism, or, in some cases, into a pair of elements.

A very familiar instance of such a change-point occurs in the quadric crank-chain when of the form known as *parallel cranks;* that is, when *a* and *c* are equal in length. and considerably shorter than *b* and *d*, which are also equal This is of course a particular case of the condition illustrated in Fig. 48, and is shown in Fig. 49.

At the instant when all the links are in a line it becomes possible for the mechanism *abcd* either to take up such a form as *abc'd'* or to continue its motion in its original form. The mechanism in question is of common occurrence in locomotive engines having coupled driving-wheels, and the necessary constraint at the change-points is provided by duplicating the chain, namely, by arranging another pair of cranks and a coupling-rod on the other side of the engine, so placed that the change-points of the two chains do not occur at the same time. Other methods of obtaining a similar object will be found discussed in a later chapter.

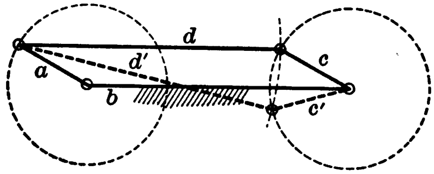


FIG. 49.

By the term *dead-point* in a mechanism is meant a position of the various links such that one of them directly opposes itself to the action of the forces tending to produce motion. The term was first applied by Watt to those positions of the crank and connecting-rod in a steam-engine in which the axes of three turning pairs lie in one plane, so that a force applied to the piston is not able to cause any torque on the shaft. It is plain that, in the absence of some means of overcoming this difficulty, the further motion of the chain becomes impossible, and the chain may be regarded as incomplete. The occurrence of dead-points must not be confused with that of change-points, although they may, and often do, occur together.

It is important to note an essential difference between dead-points and change-points. The occurrence of a dead-point depends on the particular link to which the driving force is applied, and on the manner of its application. For

instance, in the lever-crank mechanism of Fig. 42, if the crank be turned by the application of a continuous torque to its shaft, no dead-point exists. In the very same mechanism, however, if the driving effort be applied to the lever c (as in a beam-engine), dead-points occur twice in each revolution of the crank.

A change-point, on the other hand, is caused by the configuration of the chain itself, and is present whichever link is fixed, so long as the chain is the same.

31. Special Forms of Quadric Crank-chain.—Suppose in the quadric crank-chain that we make $a=b$ and $c=d$; we then obtain a chain called by Professor Sylvester the ***kite***, from its form. When one of the links is fixed, and the motion examined, it will be found that there are two change-

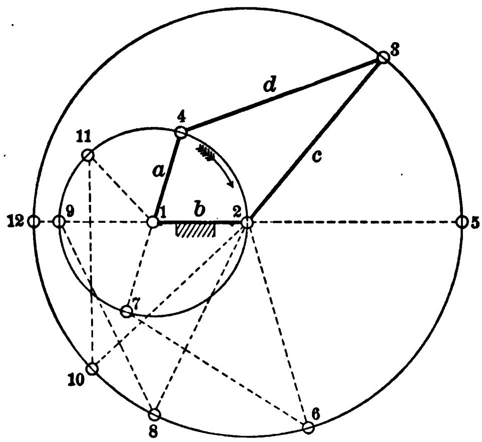


FIG. 50.

points. Thus, in Fig. 50, imagine that a rotates in the direction of the arrow, b being fixed. When the joint 4 coincides with 2 the chain becomes, for the instant, a turning pair, having its centre at 2, c and d rotating together. If a continues its motion for another complete rotation, another change-point occurs, the chain having passed through the

positions 1, 2, 6, 7; 1, 2, 8, 9; 1, 2, 10, 11; and so on, until the position 1, 2, 12 is reached. If proper restraint is applied at the change-points, so as to prevent c and d rotating together, we see that for one complete rotation of c, a will have made two revolutions.

The kite finds its principal application in certain straight-line motions,* some of which will be presently discussed.

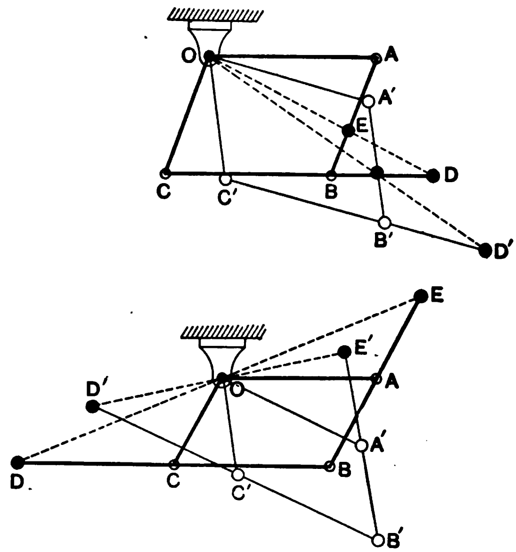


FIG. 51

A quadric crank-chain in which opposite links are of equal lengths is employed (with the addition of a fifth fixed link) for copying purposes under the name of a *pantagraph*.

Let $OABC$, in Fig. 51, represent such a chain attached to a point on a fixed link at O, and let E and D be any two points fixed on AB and CB, so that OED is originally a straight line.

* Kempe, How to Draw a Straight Line. Macmillan, Nature Series, 1877

The figure $OABC$ is a parallelogram whatever the position of the mechanism; hence the angles OAE, EBD are always equal. The lengths of the links are invariable, hence $\frac{OA}{AE}=\frac{BD}{BE}$ for any position, so that the triangles OAE, DBE are always similar, the angle OEA = the angle BED, and OED remains a straight line whatever the position of the mechanism, as at $OA'B'C'$.

If we suppose a pencil to be attached to E, any figure it describes will therefore be the polar projection of the figure described by D; the two curves being similar and similarly placed with regard to the pole O. The ratio of reduction is of course $\frac{OE}{OD}$. The fixed point O need not be at the join of the two links, but may be anywhere on any link, so long as O, E, and D are taken in one straight line.

If in the ordinary parallelogram we replace the bars AB, BC by wider pieces of material, and then choose on these pieces points P and Q such that the triangles PAB, BCQ are similar, we obtain a mechanism called by its inventor, Professor Sylvester, the *skew pantagraph* (Fig. 52).

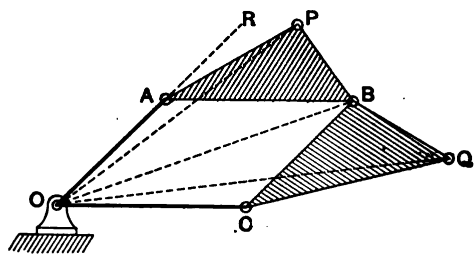


FIG. 52.

Join OP, OB, OQ. Then

$$\frac{PA}{AB}=\frac{BC}{CQ} \quad \text{and} \quad \frac{PA}{OC}=\frac{OA}{CQ}.$$

But the angles PAO, QCO are seen to be equal, therefore the triangles PAO, OCQ are similar and

$$\frac{OP}{OQ}=\frac{OA}{CQ}=\frac{AP}{OC}=\text{constant}.$$

Further, the angle $POQ = AOC - AOP - QOC$
$= AOC - (AOP + APO)$.

Produce OA to R; then

angle $POQ = RAB - RAP = PAB = \text{constant}$.

From the facts that OP and OQ are in constant ratio and include a constant angle, it is evident that the paths of P and Q are similar but of different sizes, and one is turned through the angle POQ with regard to the other.

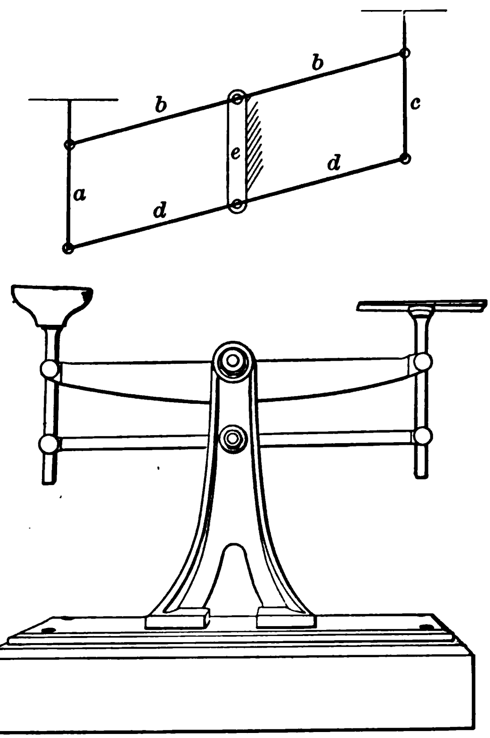


FIG. 53.

The parallelogram linkage is also applied in other ways. Fig. 53 represents the Roberval Balance in which a fifth

link is added and the parallelogram is thus doubled. During the movement of the mechanism the distance moved by a point on c is equal to that moved in the opposite sense by any point on a. Accordingly if used as a balance it does not matter where the weights are placed on the scales, and the latter remain always horizontal.

32. **Straight-line Motions.** — Under the name *straight-line motions* may be considered mechanisms whose characteristic feature is that one or more points in them travel exactly or approximately in straight lines, without being directly guided. Those contrivances which contain turning pairs only will be discussed here; nearly all of them are formed from the quadric crank-chain.

In the earlier days of mechanical engineering plane surfaces were not so readily produced as at the present time, and straight-line motions, or parallel motions as they were called, played an important part as substitutes for straight-line guides. The linkage employed by Watt is shown in its simplest form in Fig. 54, and is seen to be a quadric crank-chain so proportioned that one point on the shortest link

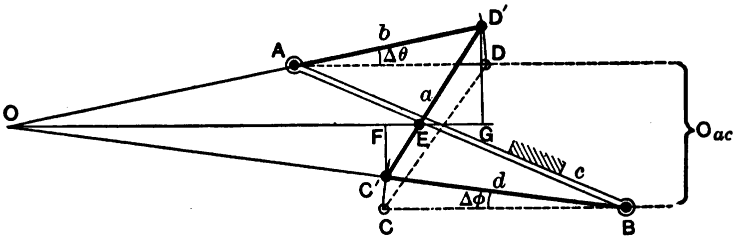


FIG. 54.

describes a path of which certain parts are approximately straight, the whole path being a kind of lemniscoid, or figure of eight. Such a mechanism is now used as a guide for the motion of the pencil of a Richards steam-engine indicator.

The mechanism consists of four links, the fixed link c being usually the longest, while b and d are generally of equal length.

This condition is not, however, necessary. Suppose the proportions to be those shown in the figure. When the mechanism is in such a position that b and d are parallel, as shown by dotted lines in the figure, the virtual centre of a with regard to c is at an infinite distance away, in a direction parallel to b and d. At that instant every point on a is moving in a direction perpendicular to its virtual radius and therefore at right angles to AD and CB. We wish to find a point on a which will describe a straight line (approximately) during small displacements of the mechanism from its dotted position. Such a straight line must be at right angles to the parallel position of b and d, as we have just seen.

Imagine that the mechanism is moved into such a position as that shown by the full lines $AD'C'B$, b describing a small angle $\Delta\theta$, while d describes a small angle $\Delta\varphi$. The position of the virtual centre O_{ac} being found at O, it is evident that the point required on $C'D'$ must be such that its virtual radius is parallel to AD or CB, for if not, it would be describing a line not perpendicular to those lines, and therefore not in a straight line with its original direction of motion. If, then, we draw a line OE parallel to AD and cutting $C'D'$ in E, E is the point required. Draw $C'F$, $D'G$, perpendicular to OE. For an indefinitely small displacement of the mechanism the link a may be considered as remaining sensibly parallel to itself, so that $DD'=CC'$ and

$$\frac{AD}{BC}=\frac{\Delta\varphi}{\Delta\theta};$$

also for such a displacement O is at so great a distance that we may write $OG=OF$ without sensible error; hence for very small displacements

$$\frac{\Delta\varphi}{\Delta\theta}=\frac{C'F}{D'G}=\frac{C'E}{D'E}.$$

Finally,
$$\frac{AD}{BC}=\frac{C'E}{D'E},$$

so that E must divide $C'D'$ in this proportion if its path is to be a straight line for small displacements. For larger displacements the path departs considerably from such a line.

It is better to proportion the lengths of the links so that DC is vertical when AD and BC are parallel; in that case the path of E will be approximately straight over a larger range of movement.

The links b and d may be placed both on the same side of a, as in Fig. 55. In this case E will be found to lie out-

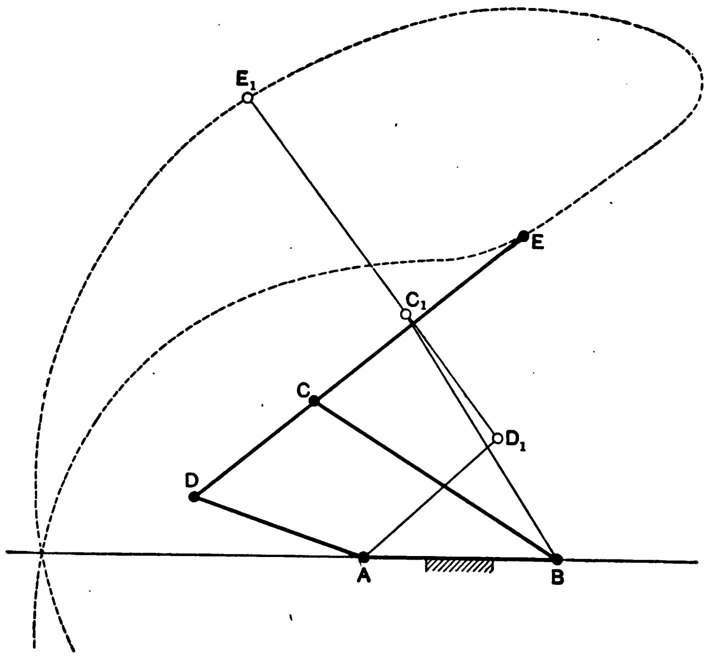


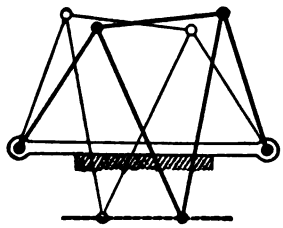
FIG. 55.

side CD, but as before $\frac{DE}{CE}=\frac{BC}{AD}$. The figure shows one half of the complete path of the point E, as well as two positions of the mechanism.

This arrangement was also used by Watt.

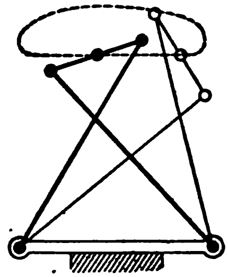
It may be shown that the part of the path of E used for a straight line is in reality wavy. Rules for designing Watt straight-line motions, as well as other forms, are given in Rankine's "Machinery and Millwork," § 253 *et seq.*

A number of other approximate straight-line motions* are derived from the quadric crank-chain. Among these the most interesting are those of Roberts and of Tchebicheff,† shown in Figs. 56*a* and 56*b*.



ROBERTS

FIG. 56*a*.



TCHEBICHEFF

FIG. 56*b*.

33. Accurate Straight-line Motions.—The first geometrically correct straight-line motion was devised in 1864 by M. Peaucellier. It is shown in Fig. 57 and is a compound chain of eight links of which

$$a=b=c=d;$$
$$e=f;$$
$$g=h.$$

The links a, b, c, d, e, f form a kite-shaped figure known as the Peaucellier cell. It has the following properties:

(1) A, D, and C must always be in one straight line.

(2) $AB^2-BC^2=AG^2-GC^2$ (since AC and BE are at right angles).

Hence $AB^2-BC^2=(AG+GC)(AG-GC)=AC.AD$.

Therefore $AC.AD$ is constant.

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* Burmester's Lehrbuch der Kinematik, § 255 et seq.

† See *Engineering*, Vol. XVI, p. 284.

Suppose the chain moved until the line AC coincides with AF. Let J be the position of C, while H is the position of D. We have

$$HA.JA=DA.CA,$$

or

$$\frac{HA}{DA}=\frac{CA}{JA}.$$

Also, $FH=FD=FA$, so that H, D, and A lie on a semicircle.

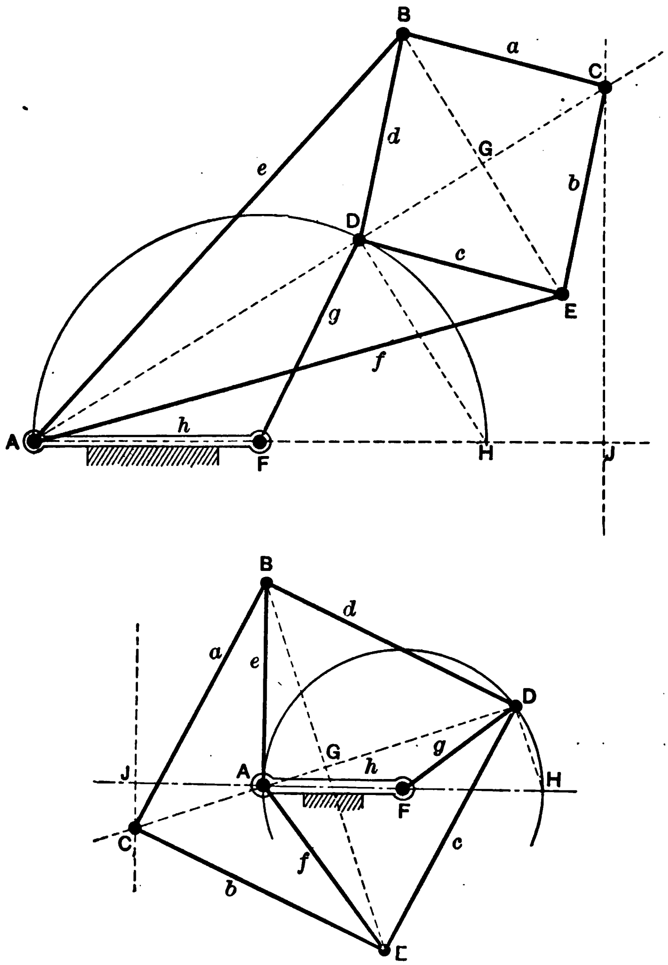


FIG. 57.

But in the triangles HAD, CAJ the angle CAJ is common, so that the triangles are similar, and the angle CJA is equal to HDA, which is the angle in a semicircle. Therefore CJA is a right angle. In the same way it may be proved that for any other position of C the line drawn from C to J is at right angles to AJ.

Hence all possible positions of C lie on a line drawn through J perpendicular to AJ, and therefore C describes a straight line.

Let us inquire what is the result of varying the lengths of the links g and h. It is plain that by making $h=0$ the points A and F would coincide and C would describe a circle of radius AC. From this it would appear probable

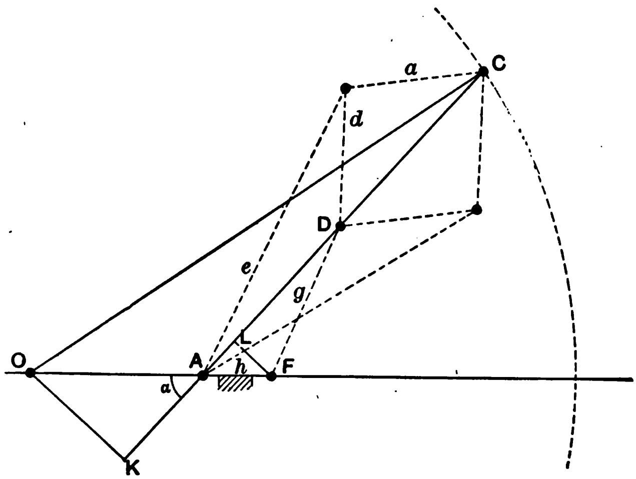


FIG. 58.

that by giving h suitable lengths less than that of g, we might describe circular arcs of any radius from AC up to infinity. If h be made greater than g, c will still describe a circular arc, but it will be convex towards A and the radius continually diminishes as g diminishes.

The locus of C may be proved to be a circular arc as follows:

In Fig. 58 we have, as before, $e^2-a^2=AC.AD$. On FA produced mark off $AO=k$, where

$$k=\frac{-h(e^2-a^2)}{h^2-g^2},$$

and

$$k(h^2-g^2)=-h(e^2-a^2).$$

Join OC. Draw OK, FL perpendicular to AC. Let the angle $CAF=\alpha$. Then

$$AD=h\cos\alpha+\sqrt{g^2-h^2\sin^2\alpha},$$

from which

$$\cos\alpha=\frac{AD^2+h^2-g^2}{2h\,.\,AD}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again,

$$AC=\sqrt{OC^2-k^2\sin^2\alpha}-k\cos\alpha,$$

from which

$$\cos\alpha=\frac{OC^2-k^2-AC^2}{2k\,.\,AC}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2),

$$\frac{AD}{h}+\frac{h^2-g^2}{h\,.\,AD}=\frac{OC^2}{k\,.\,AC}-\frac{k}{AC}-\frac{AC}{k}.$$

But
$$\frac{h^2-g^2}{h.AD}=-\frac{AC}{k}.$$

Hence
$$\frac{AD}{h}=\frac{OC^2-k^2}{k.AC}$$

and
$$OC^2=k^2-\frac{k^2}{h^2}(h^2-g^2)=\frac{g^2k^2}{h^2}.$$

But $$\frac{k}{h}=\frac{e^2-a^2}{h^2-g^2}.$$

Thus $OC=-\frac{g(e^2-a^2)}{h^2-g^2}$, which is constant.

The locus of C is therefore a circle of radius

$$R=-\frac{g(e^2-a^2)}{h^2-g^2},$$

whose centre is at a distance k from A, such that

$$k=-\frac{h(e^2-a^2)}{h^2-g^2}.$$

Note that if $h=g$, $k=\infty$ and $R=\infty$; the locus of C is then a straight line as in the Peaucellier straight-line motion. If $h<g$ and $e>a$, R is positive; while if $h>g$, R has a negative value, and the circle is convex towards A. The mechanism may thus be used conveniently for describing arcs of large circles. A graphic method of determining the proper lengths of links for this purpose has been devised by Professor Elliott.*

Another compound chain containing only turning pairs, and giving a geometrically correct straight-line motion, is that of Bricard.†

It consists of six links, arranged as in Fig. 59, such that the lengths

$$AB=AC=a;$$
$$FB=GC=b;$$
$$FG=c;$$
$$BD=CE=\frac{a^2}{b}.$$

* MacLay, Mechanical Drawing, § XLII.
† See Comptes Rendus, Vol. 120.

If the point A is to describe the straight line AK, bisecting FG at right angles, the length DE must be constant and equal to $\frac{ac}{b}$.

This is seen to be the case from the following considerations: If AF, AG, AD, AE, be joined, it is evident from

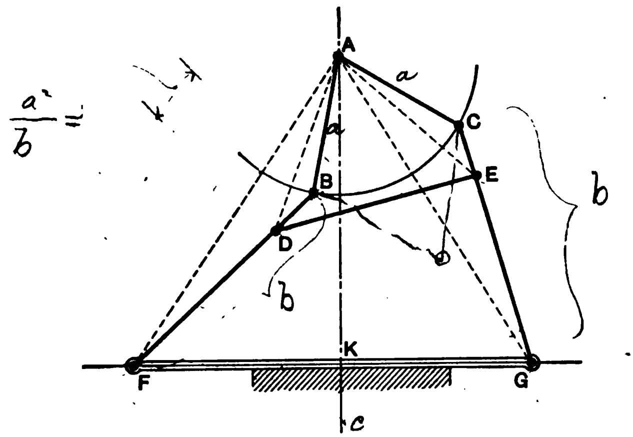


FIG. 59.

symmetry that the triangles ABF, ACG are equal in all respects, so that the angles DBA, ECA are equal. The sides AB, BD are equal to the sides AC, CE; thus $AD = AE$. Then the triangles ADF, AEG are equal and the angles DAF and EAG are equal. Add to each the angle DAG, then the angles FAG, DAE are equal. But $DA = EA$ and $FA = GA$, therefore the triangles FAG, DAE are similar and $\frac{DE}{FG} = \frac{AD}{AF}$, or $DE = FG . \frac{AD}{AF}$. But in the triangles AFB, BAD we have the angle FBA common, while by construction $\frac{FB}{AB} = \frac{b}{a} = \frac{AB}{BD}$; thus

$$\frac{FA}{AD} = \frac{FB}{AB} \quad \text{and} \quad DE = FG . \frac{AB}{FB} = \frac{ac}{b}.$$

For the required path of A, therefore, DE is constant and equal to $\frac{a}{b} \times c$.

Further information on the subject of straight-line motions will be found in the books and papers to which references have been given.

CHAPTER IV.

SLIDER-CRANK CHAINS.

34. Slider-crank Chain.—A very important chain is obtained from the quadric crank-chain by substituting a sliding pair for one of the turning pairs. It is obvious that the links will undergo the same relative change of position in Fig. 60 (*b*) as in Fig. 60 (*a*), although the lever *c* has been replaced by a block sliding in a circularly curved slot of the same radius as the original lever. The chain as thus transformed may be called a cylindric slider-crank chain, although this name is generally applied to the particular case in which O_{cd} is at an infinite distance and the block slides in a straight slot. It is plain that the mechanism of Fig. 60 (*c*) may be obtained from that of Fig. 60 (*b*) by continually increasing the radius of the pair *cd* until it becomes infinite. The pair *cd* may have prismatic surfaces of any form so long as the sliding motion is properly constrained; thus, for example, *c* may be a hollow block sliding on a prismatic rod *d*, Fig. 60 (*c*). The slider-crank chain in its cylindric form has of course plane motion, and is of special importance, since its different inversions form amongst others the mechanisms of various types of reciprocating steam-engines.

The six virtual centres of the slider-crank chain are easily found, exactly as in the case of the quadric crank-chain, but O_{cd} is always inaccessible. Fig. 61 shows the centrodes of the links *b* (representing the connecting-rod of a direct-acting engine) and *d* (representing the frame or bedplate). The centrode of *b* with respect to *d* (i.e., if *d*

is considered as the fixed link) is shown by the full line; the dotted curve represents the centrode described by O_{bd} if b

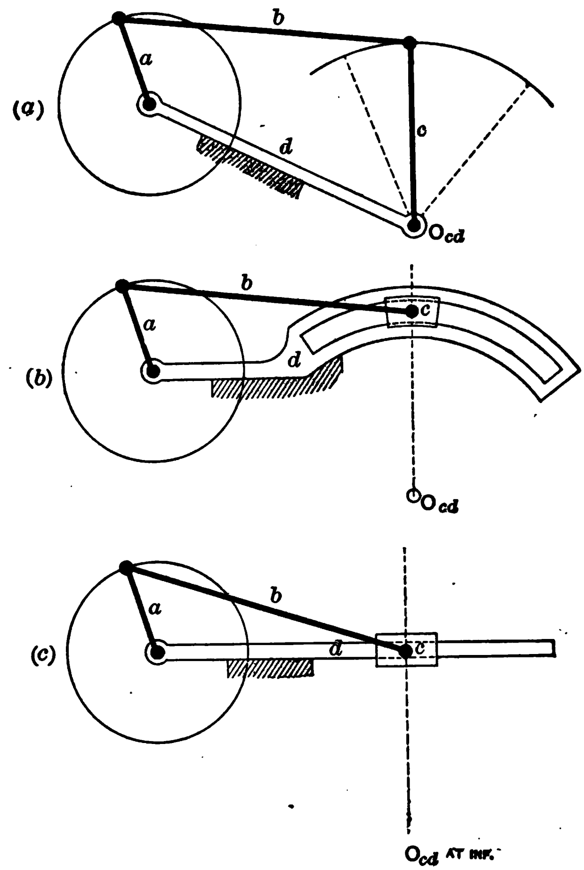


FIG. 60.

is taken as the fixed link. The construction for one point is shown in each case.

When d is fixed the link c represents the piston, piston-rod, and cross-head of the same machine. The link a represents the crank, and b the connecting-rod. A point on the link b between A and B describes an oval curve with refer-

ence to d, the shape depending on the position of the point selected, and on the ratio of the lengths of crank and con-

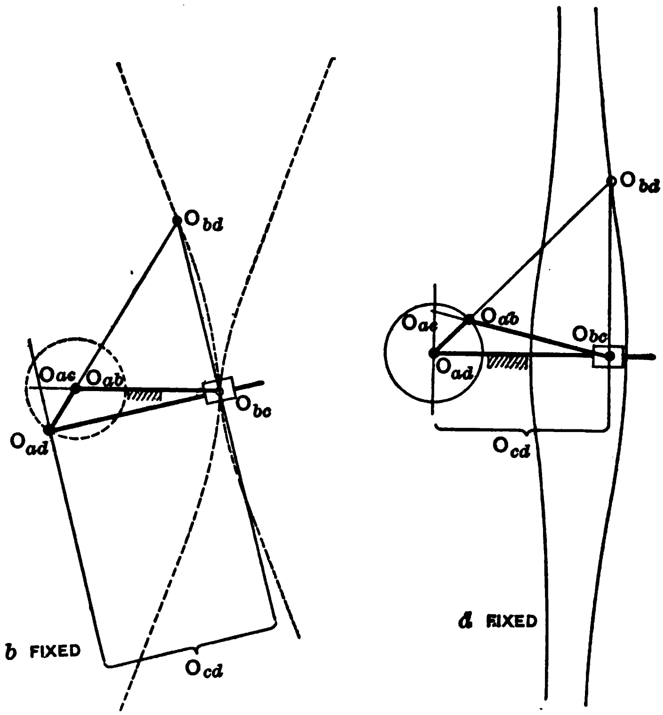


Fig. 61.

necting-rod. This fact is utilized in the design of certain valve-gears.

35. Displacement, Velocity, and Acceleration of Cross-head in Direct-acting Engine. (First Inversion of Slider-crank Chain.)—One of the most important problems in connection with the slider-crank chain is the determination of the velocity and acceleration of the link c, Fig. 60, supposing d to be fixed, and a to rotate with uniform angular velocity. This is approximately the case in a direct-acting steam-engine, where c would represent the cross-head and b the connecting-rod.

It is in general most convenient to deal with these problems graphically, but we shall first give an analytical investigation.

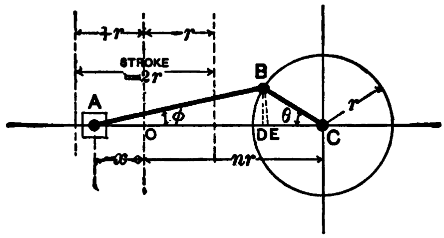


FIG. 62.

In Fig. 62, suppose the line of stroke AO to pass through C, the centre of the crank-shaft. Let BC (the throw of the crank) $=r$, and let $\frac{AB}{BC}=n$, so that the length of connecting-rod $=nr=AB$. When the crank makes any angle θ with the centre line AC, let x be the distance of the cross-head A from O, the middle of its stroke. Draw BD perpendicular to AC, and mark off $AE=AB$. If φ is the angle of obliquity of the connecting-rod,

$$\sin\varphi=\frac{\sin\theta}{n},\quad\text{and}$$

$$\cos\varphi=\frac{1}{n}\sqrt{n^2-\sin^2\theta}.$$

Now

$$\begin{aligned}x&=AC-OC=AC-AB\\&=CD+DA-AB\\&=r\cos\theta+nr\cos\varphi-nr\\&=r\,(\cos\theta-n+\sqrt{n^2-\sin^2\theta}).\quad\quad(1)\end{aligned}$$

This gives x in terms of the crank angle θ. It is to be noticed that when $\theta=\frac{\pi}{2}$ the cross-head is not at the middle of its stroke, but at a distance

$$\begin{aligned}x_0&=r(\sqrt{n^2-1}-n)\\&=-\frac{r}{\sqrt{n^2-1}+n},\end{aligned}$$

the negative sign indicating that A is now to the right of O, Fig. 62.

In the case of a cross-head having simple harmonic motion we should have simply

$$x = r\cos\theta.$$

The term $r(\sqrt{n^2 - \sin^2\theta} - n)$ in equation (1) thus gives what is called the "error due to obliquity" of the connecting-rod. Its values for $\theta = \frac{\pi}{2}$ are shown below for some usual values of n.

$n =$	4	5	6
$\sqrt{n^2 - \sin^2\theta} - n =$	-0.13	-0.11	-0.09

The error due to obliquity is thus seen to diminish rapidly as n increases.*

Next, to determine the velocity of the piston at any instant we differentiate x with regard to time and obtain

$$\frac{dx}{dt} = r\left[-\sin\theta\frac{d\theta}{dt} + \frac{1}{2}(n^2 - \sin^2\theta)^{-\frac{1}{2}}\frac{d}{dt}(n^2 - \sin^2\theta)\right].$$

$$= -r\frac{d\theta}{dt}\left[\sin\theta + \frac{2\sin\theta\cos\theta}{2\sqrt{n^2 - \sin^2\theta}}\right].$$

This is not very convenient for use in practice, but for ordinary values of n we may write without large error n instead of $\sqrt{n^2 - \sin^2\theta}$. For example, if $\theta = \frac{\pi}{2}$, and $\sin\theta$ has its greatest value,

$\sqrt{n^2 - \sin^2\theta} =$	3.87	4.89	5.91
when $n =$	4	5	6

Further, we may write V_c, the linear velocity of the crank-

* For a discussion of the problem of the connecting-rod see Hill, Min. Proc. Inst. C. E., Vol. CXXIV, p. 390. Also consult Unwin, Min. Proc. Inst. C. E., CXXV, p. 363, and a paper by G. A. Burls, Min. Proc. Inst. C. E., Vol. CXXXI, p. 338.

pin, instead of $r\frac{d\theta}{dt}$, and, omitting the negative sign, which simply shows that x diminishes at first while θ increases, we have very approximately for the velocity of the piston or cross-head

$$V_p = V_c\left(\sin\theta + \frac{\sin 2\theta}{2n}\right). \quad . \quad . \quad . \quad . \quad (2)$$

As an example, suppose an engine 12 inches stroke running at 250 revolutions per minute, the length of connecting-rod being 3 feet. The crank-pin velocity will be $\frac{250 \times 3.14}{60} =$ 13.08 feet per second. When $\theta = 45°$, the value of n being 6, we have, from equation (2),

$$\begin{aligned} V_p &= 13.08(0.70711 + 0.08333) \\ &= 13.08 \times 0.79044 \\ &= 10.340 \text{ feet per second.} \end{aligned}$$

If the velocity were calculated from the accurate expression previously obtained, we should get

$$\begin{aligned} V_p &= 13.08\left(0.70711 + \frac{1}{2\sqrt{36 - 0.4998}}\right) \\ &= 13.08 \times 0.79103 \\ &= 10.348 \text{ feet per second.} \end{aligned}$$

The approximation, therefore, has led to an error of only 0.008 foot per second in this case.

Proceeding to determine the acceleration of the piston or cross-head for any crank angle, we find very approximately from equation (2), remembering that V_c is constant,

$$\frac{d}{dt}(V_p) = V_c\left(\cos\theta\frac{d\theta}{dt} + \frac{1}{2n}\left[2\cos 2\theta\frac{d\theta}{dt}\right]\right)$$

Now $\frac{d\theta}{dt} = \frac{V_c}{r}$; thus

$$\text{acceleration of piston or cross-head} = \frac{V_c^2}{r}\left(\cos\theta + \frac{\cos 2\theta}{n}\right). \quad (3)$$

The following table gives the value of $\cos\theta + \cos\frac{2\theta}{n}$ for different values of θ and n.

θ.	Value of n.					
	4	4.5	5	5.5	6	∞
0° or 360°.......	1.250	1.222	1.200	1.182	1.167	1.000
30° or 330°.......	0.991	0.977	0.966	0.957	0.949	0.866
60° or 300°.......	0.375	0.389	0.400	0.409	0.417	0.500
90° or 270°.......	−0.250	−0.222	−0.200	−0.182	−0.167	0.000
120° or 240°.......	−0.625	−0.611	−0.600	−0.591	−0.583	−0.500
150° or 210°.......	−0.741	−0.755	−0.766	−0.775	−0.783	−0.866
180°.......	−0.750	−0.778	−0.800	−0.818	−0.833	−1.000
	Values of $\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$					

36. **Graphic Methods for Cross-head Velocity and Acceleration.**—We proceed to consider graphic means of determining velocity and acceleration for the cross-head or piston of a direct-acting engine. It is of course possible to draw first a curve of displacement on a time base, and then use the methods described in Chapter II, but simpler means can be employed in this case. In Fig. 63 let AB, BC represent the connecting-rod and crank in any given position. The point A is moving along the straight line AC, while B is moving for the instant in a direction perpendicular to BC. Hence D, the virtual centre of AB with regard to the fixed

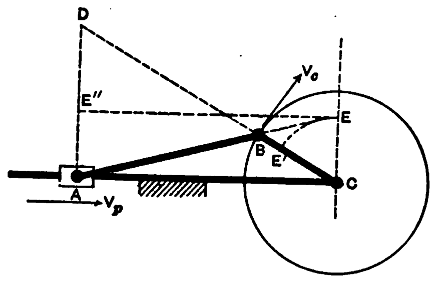


FIG. 63.

link, is easily found at the intersection of the virtual radii of the points A and B. Through C draw a line perpendicular to AC, and therefore parallel to AD, and produce AB to meet it in E. Then the triangles ADB, ECB are similar.

Now the angular velocity of AB about D is measured either by the ratio $\frac{V_p}{AD}$ or by $\frac{V_c}{BD}$, so that

$$\frac{V_p}{V_c}=\frac{AD}{BD}=\frac{CE}{CB}.$$

In many positions of the mechanism D is inaccessible, but E can always be found, and the relation just obtained tells us that CE *represents the velocity of the piston* at the instant for which the diagram is drawn, to the same scale as that to which CB represents the velocity of the crank-pin.

It is generally most convenient to make a polar diagram of piston velocity by marking off a series of points such as E' (where $CE'=CE$) for a number of different crank positions, or, if required, a velocity diagram on a distance base may be constructed by marking off the distance CE along AD, so that a series of points such as E'' are obtained, and a curve drawn whose ordinate at any point is proportional to the velocity of the piston when in that position. Such diagrams have been drawn in Fig. 64, together with a linear velocity diagram on a time base, so as to show the difference between a simple harmonic motion and that which the piston actually possesses. The example taken is that for which the velocity and acceleration have been calculated in the last section. In order to determine the scale to which the ordinates of the curves represent the velocity, it is only necessary to remember that if the length BC were 1 inch, the velocity scale would be 1 inch = 13.08 feet per second, since the crank-pin velocity is 13.08 feet per second. In the figure the construction lines are shown for one position of the mechanism only; in drawing such diagrams care should

be taken only to draw those portions of the construction lines which are absolutely necessary, so as to avoid useless complication. Of course accuracy in drawing is indispensable if the numerical results obtained are to be reliable. A line whose length is proportional to the *piston acceleration*

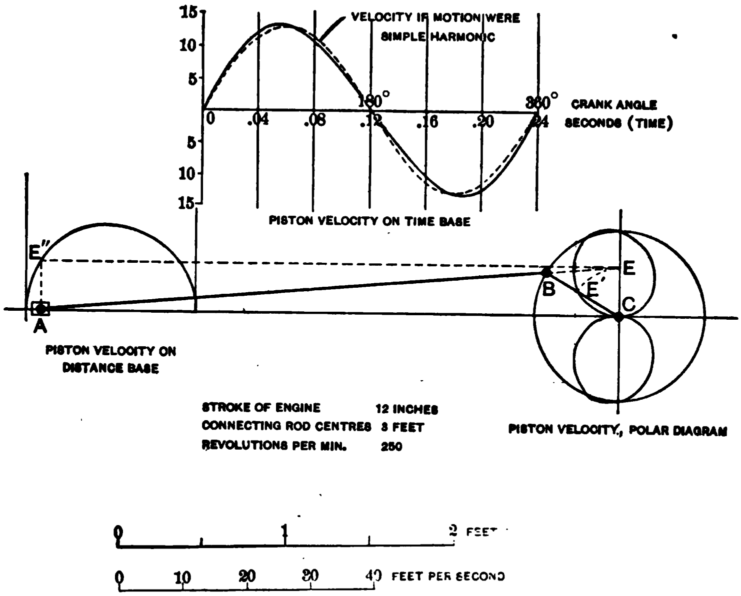


FIG. 64.

may be obtained as follows (see Fig. 65): Take any given position of the cross-head A, and produce AB to cut CE in E. Then, as before,

$$\frac{CE}{CB}=\frac{V_p}{V_e}.$$

Notice that this is true whether the path of the point A passes through C or not, when produced.

The desired acceleration is the rate of change of V_p, which is of course proportional to the rate at which the distance CE is increasing or diminishing at the instant considered. In fact the piston *acceleration* may be considered

as being proportional to the *velocity* of the point E along CE at any instant while the engine is in motion, supposing BE always to be in a straight line with AB.

Let this velocity along CE be u_0. The real velocity of the point E, regarded as a point on the connecting-rod, is in a direction perpendicular to DE, its virtual radius. Calling this velocity u_1, we see that u_1 may be resolved into two

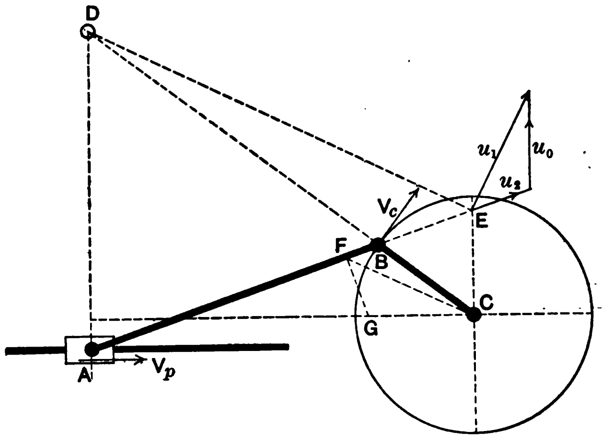


FIG. 65.

components, namely, u_0 in a direction along CE, and u_2 in a direction along BE.

From C draw CF parallel to DE, and draw FG perpendicular to AB. Then the sides FC, CG, GF, of the triangle FCG are respectively perpendicular to the directions of u_1, u_0, u_2. Thus FCG is a triangle of velocities and $\frac{u^0}{u_1}=\frac{CG}{FC}$, or $u_0=u_1\frac{CG}{FC}$. But $\frac{u_1}{V_c}=\frac{DE}{DB}=\frac{FC}{CB}$, therefore $u_1=V_c\frac{FC}{CB}$, and

$$u_0=V_c\frac{CG}{CB}=\text{rate of change of length } CE.$$

Now it has been shown that

$$\text{piston velocity}=V_p=CE.\frac{V_c}{CB},$$

and V_c and CB are constant; hence it follows that the rate of change of the piston velocity must be equal to

$$\text{(rate of change of } CE) \times \frac{V_c}{CB}, \text{ that is,}$$

$$\text{piston acceleration} = u_0 \frac{V_c}{CB} = \frac{V_c^2}{CB^2} . CG.$$

Thus to obtain the numerical value of the piston acceleration we must multiply the length of CG (measured to scale in feet) by $\left(\frac{V_c}{r}\right)^2$, where V_c is the crank-pin velocity in feet per second and r is the crank throw, or radius of the crank-pin circle, in feet.

Hence it follows that

$$\frac{CG}{CB} = \frac{\text{acceleration of piston}}{V_c^2/r},$$

or, in other words, CG represents the piston acceleration to the same scale as that on which CB represents V_c^2/r, the radial acceleration of the crank-pin.

When drawing such a diagram as Fig. 65 it happens that for many positions of the crank the point D becomes inaccessible. Accordingly some other construction must be found to obtain the position of the point F, so that CG may be determined for any crank angle.

Consider the triangles BEC and BAD.

Evidently

$$\frac{BE}{BA} = \frac{BC}{BD}.$$

But $\frac{BC}{BD} = \frac{BF}{BE}$, because the triangles BDE, BCF are similar.

Therefore

$$\frac{BE}{BA} = \frac{BF}{BE},$$

or

$$BA . BF = BE^2.$$

Hence any construction which will make BE a mean proportional between BA and BF will determine the point F.

A number of such constructions have been given; of these perhaps the most convenient in practice is that of Kisch.*

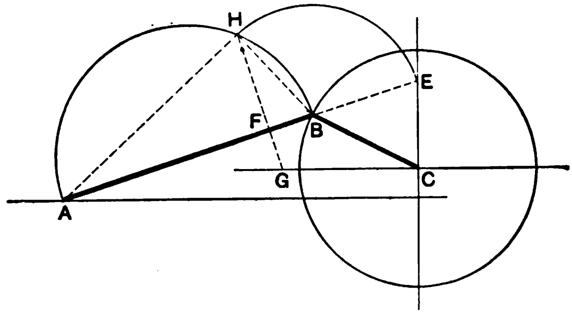


FIG. 66.

On AB describe a semicircle AHB. With centre B and radius BE cut the semicircle in H. Draw HFG perpendicular to AB, cutting AB in F and CG in G. Join BH, HA. Then

$$\frac{BF}{BH}=\frac{BH}{BA}.$$

But $BE=BH$. Hence $BA.BF=BE^2$, and CG represents the acceleration.

The method of determining the acceleration scale of such a diagram may be shown by a numerical example. Fig. 67 has been drawn for the engine for which the velocity of the piston has been previously calculated, taking a crank angle of 45°. The crank-pin velocity being 13.08 feet per second, and the connecting-rod being 6 cranks in length, we have for the acceleration of the piston at that particular crank angle

$$\begin{aligned}\text{acceleration} &= \frac{13.08^2}{0.5}\left(\cos 45^\circ + \frac{\cos 90^\circ}{6}\right)\\ &= \frac{13.08 \times 13.08 \times 0.70711}{0.5}\\ &= 242.1 \text{ feet per second per second.}\end{aligned}$$

* See Zeitschrift des Vereines Deutscher Ingenieure, Dec. 13, 1890. Given also by Klein, Journal of Franklin Inst., Vol. CXXXII, Sept. 1891.

In Fig. 67 the actual length of the line CG, if the figure were drawn the full size of the engine, would be 0.351 foot. The radius of the crank-pin circle CB is 0.5 foot and represents a velocity of 13.08 feet per second. Hence the velocity scale is 1 foot = 26.16 feet per second, or $\frac{V_c}{CB}=\frac{13.08}{0.5}=26.16$.

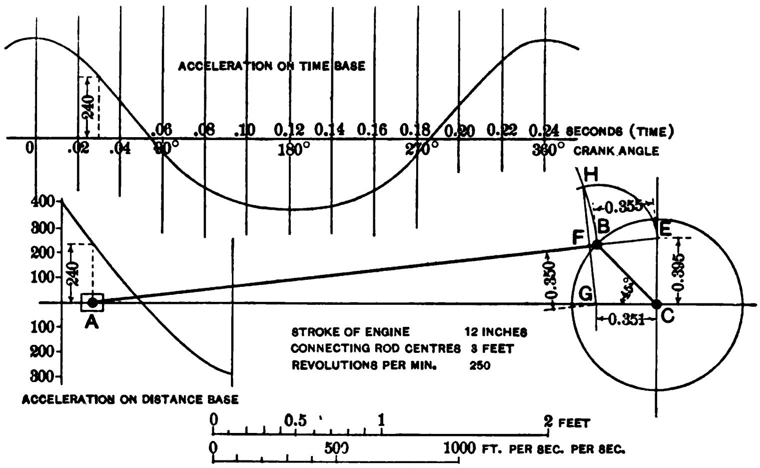


FIG. 67.

It has been shown that the rate at which the distance CE is changing is

$$u_0=V_c\cdot\frac{CG}{CB}$$

$$=0.351\times 26.16=9.18 \text{ feet per second.}$$

This, however, is not the numerical value of the acceleration required, for it represents the rate of change of a length CE, each foot of which stands for a velocity of 26.16 feet per second. Therefore, expressing u_0 in feet per second per second, we have

$u_0=0.351\times 26.16\times 26.16=240$ feet per second per second, a result agreeing (within the limits of accuracy for a small scale drawing) with that just calculated.

It is thus seen that to determine the scale to which CG

represents the piston acceleration, we find, first, the piston velocity represented by unit length of CE (in this case 26.16 feet per second); then it follows that a change of length of CE at the rate of one unit per second represents a change of piston velocity at the rate of 26.16 units per second, or a piston acceleration of 26.16 units. But each unit of length of CG has been shown to represent a change of length of CE at the rate of 26.16 units per second, so that, finally, unit length of CG represents a piston acceleration of 26.16×26.16 units.

This relation may be expressed by saying that if the engine were drawn out full size and the linear velocity scale were 1 foot $= n$ feet per second, then the acceleration scale would be 1 foot $= n^2$ feet per second per second. In this case, as in the case of all graphic methods of determining velocities and accelerations, the manner of finding the velocity and acceleration scales must be thoroughly understood; if this is not done, the diagram becomes almost useless, since no numerical values can be obtained from it.

A number of other constructions for the piston acceleration in the direct-acting engine have been devised.*

37. Angular Velocity and Acceleration of Connecting-rod.—To study the movement of the connecting-rod, adopting the same notation as in § 35, we have, as before,

$$\sin \varphi = \frac{\sin \theta}{n},$$

$$\cos \varphi = \frac{\sqrt{n^2 - \sin^2 \theta}}{n}.$$

The angular velocity of the connecting-rod is the rate of change of φ with regard to time, and we obtain at once

$$\frac{d\varphi}{dt} = \frac{\cos \theta}{\sqrt{n^2 - \sin^2 \theta}} \frac{d\theta}{dt}.$$

* See a paper by Prof. Elliott, Engineering, Vol. LIX, pp. 587 and 711, and Zeitschrift des V. Deutscher Ingenieure, Oct. 13, 1894.

Since $\frac{d\theta}{dt}$ is the angular velocity of the crank, we have

$$\text{angular velocity of connecting-rod} = \frac{V_c}{r} \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}.$$

Differentiating again to find the angular acceleration, we obtain

$$\frac{d^2\varphi}{dt^2} = \frac{V_c}{r} \cdot \frac{d}{d\theta} \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}} \cdot \frac{d\theta}{dt}$$

$$= \frac{V_c^2}{r^2} \left\{ \cos\theta \frac{d}{d\theta}(n^2 - \sin^2\theta)^{-\frac{1}{2}} - \sin\theta (n^2 - \sin^2\theta)^{-\frac{1}{2}} \right\}$$

$$= -\frac{V_c^2}{r^2} \cdot \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}} \cdot \frac{n^2 - 1}{n^2 - \sin^2\theta}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For ordinary values of n it is sufficiently accurate to write approximately

$$\text{angular acceleration} = -\frac{V_c^2}{r^2} \cdot \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}. \quad . \quad (2a)$$

Taking the same example as before, at a crank angle of 45° we have

$$\sin\theta = 0.70711, \quad n = 6, \quad V_c = 13.08, \quad r = 0.5.$$

Thus $\sqrt{n^2 - \sin^2\theta} = 5.96$ and $\cos\theta = 0.70711$. Therefore

$$\text{angular velocity} = \frac{13.08 \times 0.70711}{0.5 \times 5.96} = 3.11 \text{ radians per second,}$$

and, from equation (2),

$$\text{angular acceleration} = -\frac{13.08^2}{0.25} \cdot \frac{0.70711}{5.96} \cdot 0.987$$

$$= -80.2 \text{ radians per second per second.}$$

Using equation (2a), we should obtain -81.2 as a result.

The simple construction of Figs. 65, 66, and 67 gives us the angular velocity of the connecting-rod. For

$$\text{angular velocity of crank} = \frac{V_c}{BC},$$

and $$\text{angular velocity of connecting-rod} = \frac{V_c}{BD}.$$

But $\frac{BC}{BD}=\frac{BE}{BA}$; hence

$$\text{angular velocity of connecting-rod} = \frac{V_c}{r}\cdot\frac{BE}{BA}.$$

If BE is taken at the length it would have were the engine drawn out full size, $BA = nr$, and

$$\text{angular velocity} = BE\cdot\frac{V_c}{nr^2}.$$

(Note that the real lengths of BE and r must be used, measured in feet, V_c being in feet per second.)

In Fig. 67, for example, BE scales 0.355 foot, hence the angular velocity will be

$$0.355\times\frac{13.08}{6\times 0.25}=3.10 \text{ radians per second,}$$

a result agreeing with the calculated value.

As regards the angular acceleration we have seen that

$$FC : CG : GF :: u_1 : u_0 : u_2.$$

The velocity u_2 is the rate at which the length BE is changing, and is therefore proportional to the rate of change of the angular velocity of the connecting-rod. Hence it may be shown (just as in the case of the velocity u_0) that

$$\text{angular acceleration} = FG\cdot\frac{V_c^2}{nr^3}.$$

In our example (Fig. 67) FG is 0.350 feet; hence

$$\text{angular acceleration} = 0.350\times\frac{13.08\times 13.08}{6\times 0.125}$$

$$= 79.9 \text{ radians per second per second,}$$

a result agreeing closely with that previously obtained.

Notice that when the crank angle is 90° FG becomes $\frac{nr}{\sqrt{n^2-1}}$ and therefore, if $\theta=\frac{\pi}{2}$,

$$\text{angular acceleration} = \frac{V_c^2 . nr}{nr^3\sqrt{n^2-1}}=\frac{V_c^2}{r^2\sqrt{n^2-1}}.$$

38. Angular Velocity of Cylinder in Oscillating Engine. Second Inversion of Slider-crank Chain.—The second inversion of the slider-crank chain is that in which the link b

(represented by the connecting-rod in a direct-acting steam-engine) is the fixed link. This mechanism is known as the swinging-block slider-crank and is employed as an oscillating steam-engine, of which the link d becomes the piston and rod, while b is the framework. The link c is the cylinder and a is the crank, the cylinder swinging to and fro on trunnions as the crank-shaft revolves. We proceed to compare the angular velocity of the cylinder with that of the crank, the latter being supposed to rotate uniformly.

Let Fig. 68 represent this mechanism. The distance AB is the length of the fixed link, measured from the centre of the cylinder-trunnions to the centre of the crank-shaft, while BC is the half-stroke of the piston. Let $\frac{AB}{BC}=n$. Let the angle the crank has turned through from its lowest position be θ, φ being the angle at which the centre line of the cylinder is inclined to AB. Then

$$\tan\varphi=\frac{CD}{DA}=\frac{\sin\theta}{n-\cos\theta}.$$

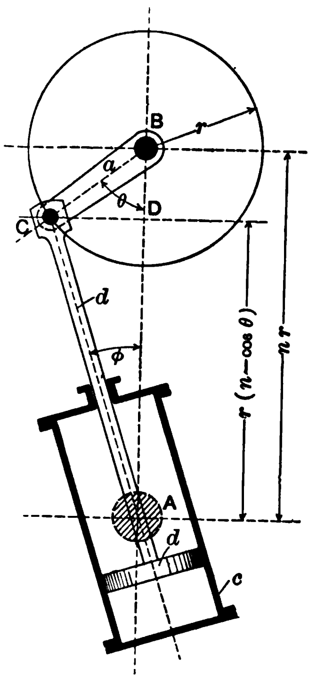


FIG. 68.

The angular velocity of the cylinder is $\frac{d\varphi}{dt}$, so that on differentiating

$$\frac{d\varphi}{dt}=\cos^2\varphi\cdot\frac{d}{d\theta}\frac{\sin\theta}{n-\cos\theta}\cdot\frac{d\theta}{dt}$$

$$=\cos^2\varphi\,\frac{n\cos\theta-1}{(n-\cos\theta)^2}\cdot\frac{d\theta}{dt}.$$

But $\cos^2\varphi=\frac{AD^2}{AC^2}=\frac{AD^2}{AD^2+DC^2}=\frac{(n-\cos\theta)^2}{(n-\cos\theta)^2+\sin^2\theta}.$

Thus $\frac{d\varphi}{dt}=\frac{n\cos\theta-1}{n^2-2n\cos\theta+1}\cdot\frac{d\theta}{dt}.$

From this we find by again differentiating

$$\frac{d^2\varphi}{dt^2} = -\frac{n \sin\theta(n^2-1)}{(n^2-2n\cos\theta+1)^2}\cdot\left(\frac{d\theta}{dt}\right)^2,$$

which is the value of the angular acceleration of the cylinder for any crank angle θ.

Notice that since the angular velocity of the crank is uniform, the cylinder executes its forward and backward swings in unequal times. By assigning suitable proportions this particular inversion of the slider-crank chain may be utilized as a quick-return motion (see Fig. 72), by

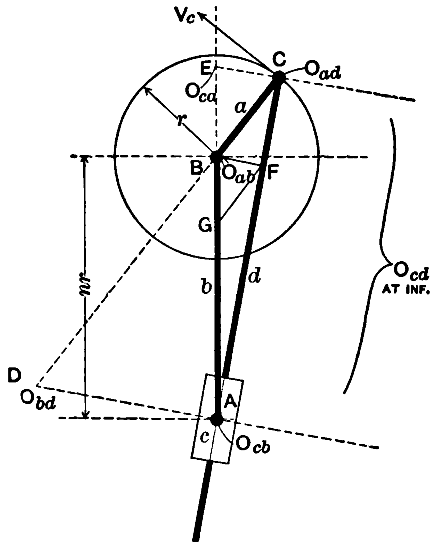


FIG. 69.

causing the swinging link c to actuate the tool-box, say, of a shaping-machine, which can thus be made to perform its return or non-cutting stroke at a quicker rate, and in less time, than its forward or cutting stroke.

The velocity ratio of cylinder and frame may readily be obtained graphically. The positions of the six virtual centres of the mechanism are shown in Fig. 69. Let ω_{ab} represent the angular velocity of the crank with respect to

the frame b; we wish to find ω_{cb}. E (the virtual centre of c with regard to a) is a point common to the two bodies a and c for the instant considered. Its linear velocity may be expressed either as $\omega_{cb} \times AE$ or as $\omega_{ab} \times EB$. Hence

$$\frac{\omega_{cb}}{\omega_{ab}} = \frac{EB}{AE}.$$

Draw BF parallel to CE, and FG parallel to CB. Then

$$\frac{\omega_{cb}}{\omega_{ab}} = \frac{EB}{AE} = \frac{CF}{AC} = \frac{BG}{BA};$$

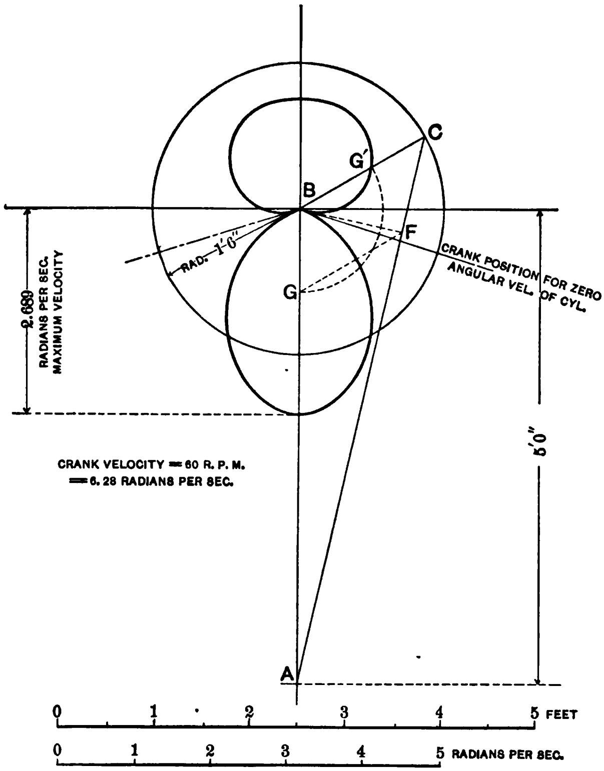


FIG. 70.

and since BA is constant, the length BG is proportional to the angular velocity of the cylinder. A polar diagram may

be drawn by marking a distance BG' along BC equal to BG and repeating the construction. This has been done in Fig. 70 for an oscillating engine in which the stroke is 3 feet and the distance AB is 5 feet. At 60 revolutions per minute the maximum angular velocity of the cylinder is represented on the diagram by a distance measuring 2.14 feet to scale, hence its numerical value (the angular velocity of the crank being 2π radians per second) will be

$$6.28 \times \frac{2.14}{5} = 2.689 \text{ radians per second.}$$

The angular acceleration may easily be obtained by construction from the velocity diagram, as is shown in Fig. 71. The value of the velocity and acceleration should be calculated for one or two positions of the crank, as an exercise, and compared with the diagrams.

Notice that the value of the angular-velocity ratio when $\theta = 0°$ or $180°$ is

$$\frac{\omega_{cb}}{\omega_{ab}} = \frac{V_c}{r(n \pm 1)} \cdot \frac{r}{V_c} = \frac{1}{n \pm 1}.$$

Notice also that the angular velocity and angular acceleration will be the same for the piston and rod as for the cylinder. It is easy to show that $\frac{\omega_{db}}{\omega_{ab}} = \frac{CB}{CD} = \frac{BG}{AB}$ (Fig. 69).

Fig. 71 shows the angular velocity and acceleration (in the same example) as plotted on a polar diagram (the acceleration curve being found from the velocity curve as in § 22), and also shows the corresponding linear diagrams on a time base, the scales being marked. The linear diagram of acceleration could be obtained from the velocity-time curve by the method of § 19, but the acceleration scale would not then be the same as that shown on the figure.

A number of problems dealing with velocity ratios and accelerations in the oscillating engine have been worked

out by Professor Elliott in a communication to *Engineering*, Vol. LXIII, page 665, to which the reader is referred.

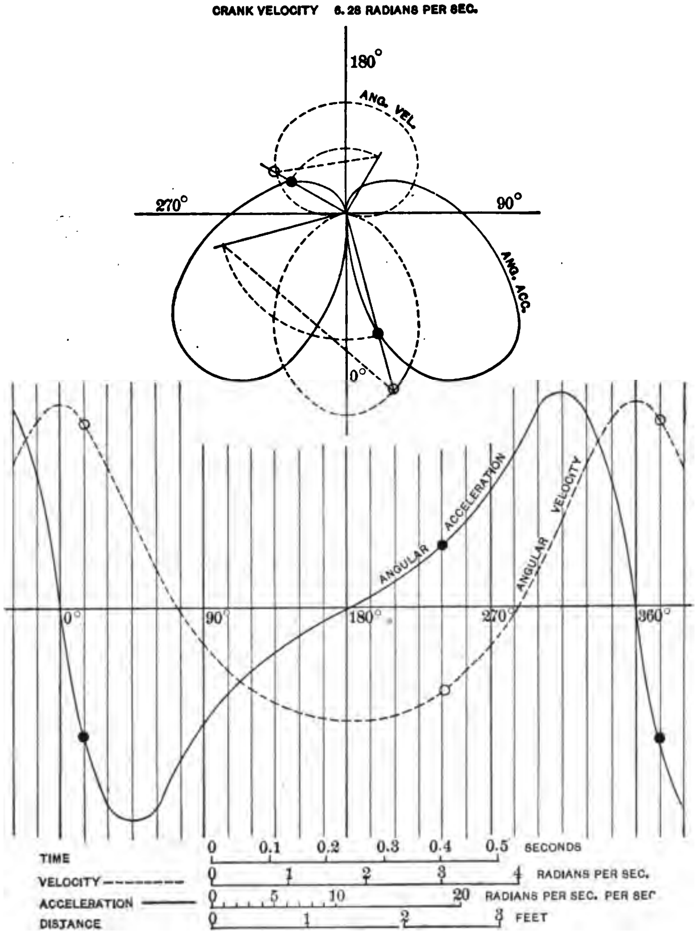
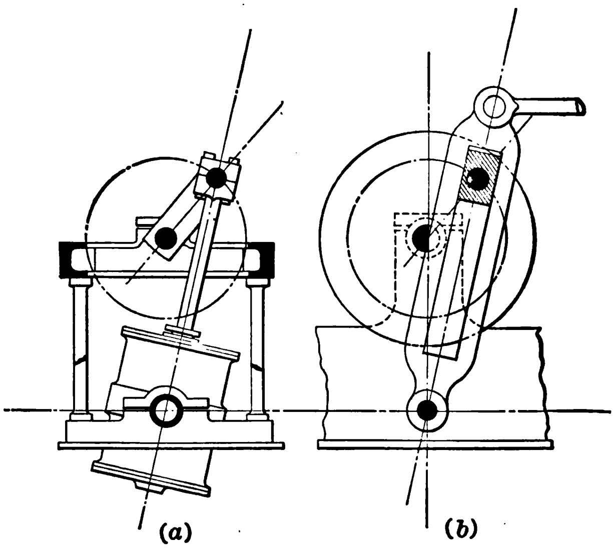


FIG. 71.

A general method will be given later (Chapter V) by which the linear or angular velocity or acceleration may be found graphically for any point on a link of a mechanism of the kind discussed in this chapter.

A good example of the kinematic identity of mechanisms

which at first sight appear to be very different is afforded in Fig. 72. The links which correspond in the two cases have the same letters attached. The sketch (*a*) represents the oscillating engine, while (*b*) gives a diagrammatic view of the corresponding quick-return motion. Both are derived



(*a*) (*b*)

FIG. 72.

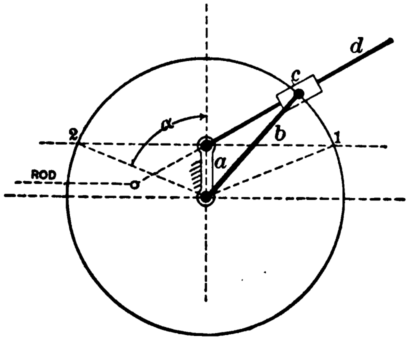
from the same inversion of the slider-crank chain. The swinging link *c* has the same relative motions in regard to the links *b* and *d* (with which it pairs) in the quick-return motion as in the engine. The framing of the engine corresponds to the fixed framework of the machine-tool. The rod *R*, which of course does not appear in the engine, communicates the variable motion of the swinging link *c* to the tool-carriage. The crank-shaft of the engine is represented by the disc *a*, to which rotary motion is imparted by the driving mechanism of the tool (not shown).

39. Whitworth Quick-return Motion. Third Inversion of Slider-crank Chain.—Passing to the next inversion, we now have *a* as the fixed link, and the resulting mechanism

is one which was applied by Whitworth* as a quick-return motion for the same purposes as have already been mentioned. It has been called by Reuleaux the *turning-block slider-crank chain.*

The velocity ratio of the links b and d may be obtained

FIG. 73*a*.



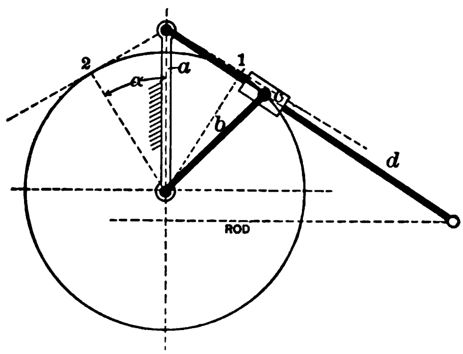


FIG. 73*b*.

exactly as in the case of the oscillating engine. In fact an alteration in the relative lengths of the links a and b, Fig. 70, converts the mechanism there shown into one not differing in any essential particular from the quick-return motion of Fig. 73*b*.

* See Cotterill, Applied Mechanics, p. 108, § 50.

Figs. 73*a* and 73*b* show this quick-return motion with different proportions of the lengths of the links *a* and *b*. It will be seen that while *d* executes complete rotations in the first case, it only swings in the second case.

The relative angular velocities of *b* and *d* are easily found by the construction shown in Fig. 69. Evidently if *a* is the fixed link

$$\frac{\omega_{da}}{\omega_{ba}} = \frac{BD}{CD} = \frac{BA}{EA} = \frac{GA}{BA}.$$

Thus if AB represents the angular velocity of the link *b*, AG represents that of *d* to the same scale.

Supposing the link *b* to have a uniform angular velocity, the positions 1 and 2 are those in which the tool-box is at one or the other end of its travel. Accordingly it is easily seen that the times of the cutting and return strokes will be in the ratio $\frac{180-\alpha}{\alpha}$. Hence in designing such a motion we have only to proportion *a* and *b* so that $\frac{a}{b} = \cos\alpha$ in the Whitworth motion, or $\frac{b}{a} = \cos\alpha$ in the other form, where α has such a value as to make $\frac{180^\circ-\alpha}{\alpha}$ the desired ratio. The times of the cutting and return strokes are often in the ratio 2 : 1 or 3 : 1.

The centrodes for the links *c* and *a* are found by similar constructions to those already shown in Fig. 61 for the links *b* and *d*, and are drawn in Fig. 74. The reader should construct them for himself for the case shown (in which the length of the link *a* is less than that of *b*), and also for the case in which the link *a* is longer than *b*, when the centrodes take quite different forms.

40. Pendulum Pump. Fourth Inversion of Slider-crank Chain.—The last of the four possible inversions of the chain, the *swinging slider-crank*, in which *c* is the fixed link, has

only a very limited application in practice, but has been employed as a small steam donkey-pump. It is shown diagrammatically in Figs. 75*a* and 75*b*, and is shown also in outline in the second sketch in Fig. 74.

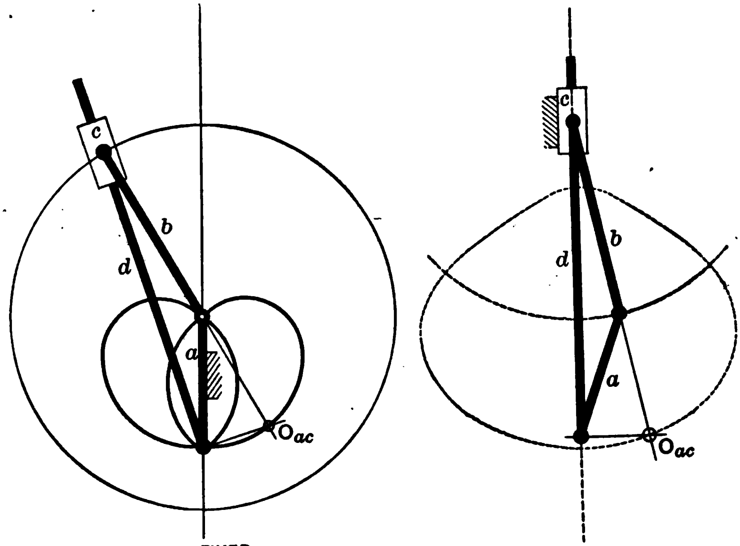


FIG. 74.

It is known as the "pendulum pump," from the motion of the link b. The link c now represents the cylinders (steam and water) and their connecting framework, while d is the piston, rod, and plunger. The crank a takes the form of a small fly-wheel, which rotates about O_{ab}, while that point swings along the dotted arc. The relative angular velocity of b and d is easily found graphically, the virtual centres being known. Let V = linear velocity of d; then, since the link a is turning for the instant about E, we have

$$\text{angular velocity of } a = \frac{\text{linear velocity of point } B}{BE} = \frac{V}{CE},$$

and

$$\text{linear velocity of point } B = V \cdot \frac{BE}{CE}.$$

Therefore

$$\text{angular velocity of } b = V \cdot \frac{BE}{BA \times CE}$$
$$= V \cdot \frac{BA}{BA \times AD} = \frac{V}{AD}.$$

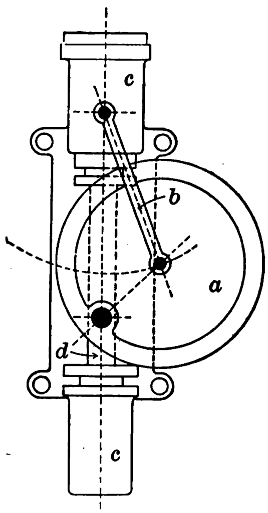


FIG. 75*a*.

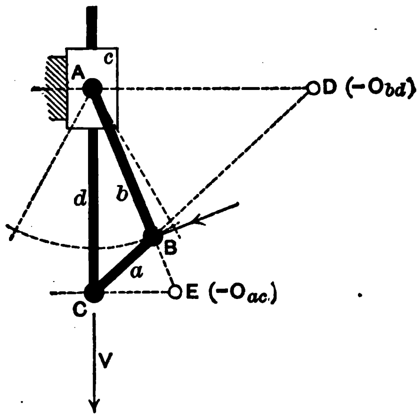


FIG. 75*b*.

The various inversions of the slider-crank chain may be summarized thus:

Link Fixed.	Name of Chain.	Resulting Mechanism.
d	Turning slider-crank	Direct-acting engine
b	Swinging-block slider-crank	Oscillating engine. Quick-return motion
a	Turning-block slider-crank	Whitworth quick-return motion
c	Swinging slider-crank	Pendulum pump

41. Crossed Slider-crank Chains.—The slider-crank chains hitherto discussed have been arranged so that the straight line in which O_{bc} moves relatively to d passes through O_{ad}. If this is not the case, we obtain a further series of mechanisms known as crossed slider-crank chains, shown in Fig. 76.

The crossed turning slider-crank has been used in certain single-acting high-speed steam-engines, with a view of lessening the effect of the obliquity of a short connecting-rod during the working stroke; the obliquity during the return stroke is of course correspondingly increased. In such a case the determination of the acceleration or velocity of the

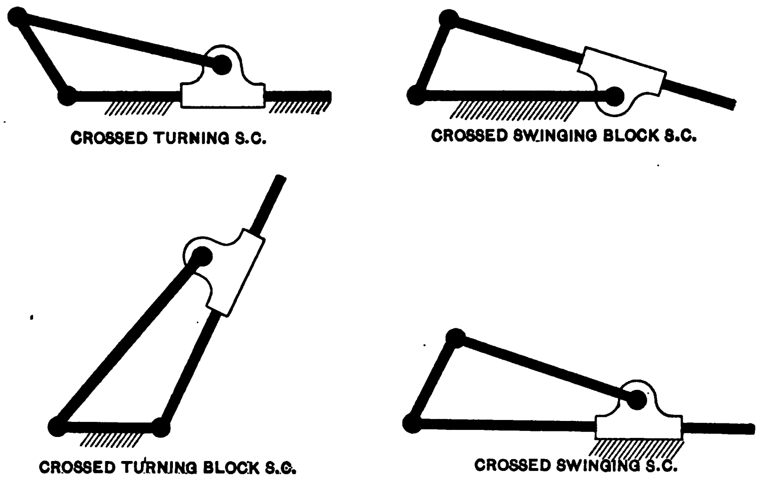


FIG. 76.

piston does not present any difficulty, as it can be carried out by the constructions already given.

42. **Double Slider-crank Chain.**—We consider next the simple chain formed by two turning and two sliding pairs. It has been already shown that from the quadric crank-chain the slider-crank chain may be derived by substituting a sliding pair for one of the turning pairs, such sliding pair being equivalent to a turning pair of infinitely great radius. This substitution may be repeated, and Fig. 77 shows the result in the case where the directions of motion of the two sliding pairs are at right angles, and where one link carries an element of each of the two sliding pairs. Such a chain is called a *double slider-crank chain*. The link b has now become compressed into a block sliding in a slot formed in c.

Since the relative motion of b and c is the same as if the pair bc were a turning pair of infinite radius, the velocity ratios and accelerations in this chain will all be found exactly as in the case of a slider-crank chain in which the link b is of

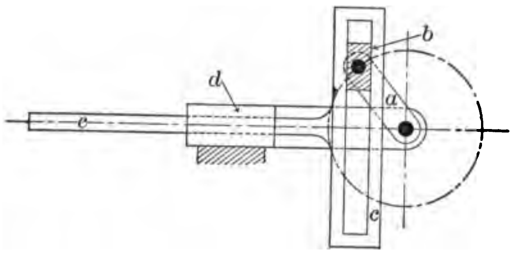


FIG. 77.

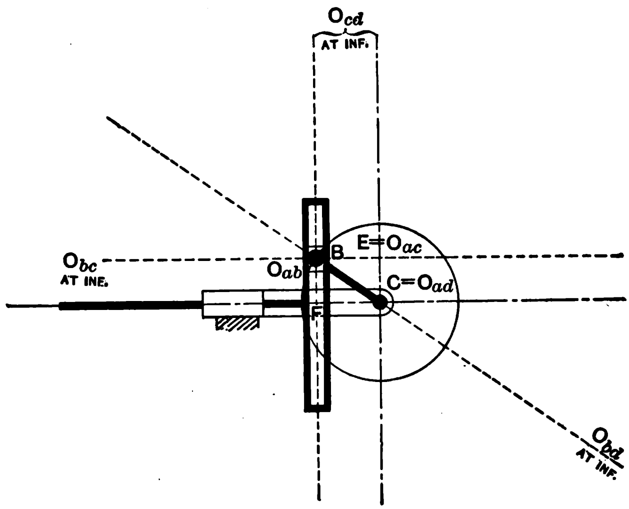


FIG. 78.

infinite length, i.e. when $n = \infty$. Accordingly we may write at once (supposing the link d to be fixed), with the same notation as before,

$$x = r \cos \theta,$$

and

$$\text{linear velocity of } c = V_p = V_c \sin \theta,$$

while

$$\text{acceleration of } c = \frac{V_c^2}{r} \cos\theta.$$

Notice that if the crank rotates uniformly the motion of *c* with regard to *d* is simple harmonic, and that the link *b* has no angular velocity with regard to *d*.

Fig. 78 shows the six virtual centres of the chain. It is plain that the centrodes of *b* and *d* are now altogether inaccessible.

The distance *CE* is seen to be proportional to the linear velocity of the link *c*, while *CF* is proportional to its linear acceleration as given above; the scales are readily determined.

This inversion of the double slider-crank chain is frequently employed in the construction of steam-pumps. The link *c* represents the steam-piston and pump-plunger, *d* the cylinder, framing, and pump-barrel, and *a* the crank-shaft. The total height of the pump may be made small, on account of the absence of a connecting-rod, thus making the arrangement a very convenient and compact one for certain purposes.

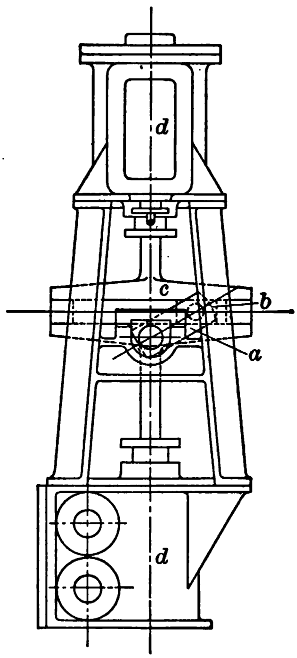


FIG. 79.

The linear and polar diagrams of piston displacement, velocity, and acceleration, supposing that the angular velocity of the crank-shaft is uniform, are precisely those already given for simple harmonic motion.

If the link *b* be supposed fixed instead of *d*, the resulting mechanism is the same as before, for the reason that the relative motion of *b* and *a* is exactly the same as that of *d* and *a*. Thus on fixing *b* we still have the link *c*

executing simple harmonic vibrations as the crank rotates uniformly, but the direction of motion is now along the line FB instead of FC (Fig. 78).

43. **Elliptic Trammels.**—If c, the link containing one element of each of the two sliding pairs, is the fixed link, we obtain a mechanism used for the purpose of drawing ellipses, and shown in Fig. 80. The bar a, carrying an element of each of the two turning pairs, now carries a movable tracing-point; the blocks b and d slide in a pair of grooves intersecting at right angles and formed in the link c.

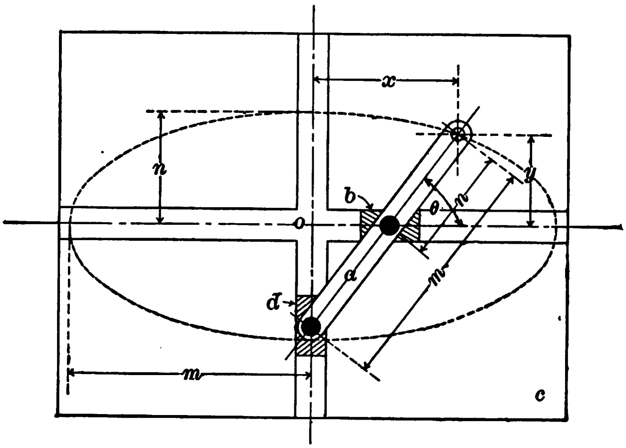


FIG. 80.

The path of the tracing-point is easily seen to be an ellipse, for, with the notation of Fig. 80, we have

$$\sin\theta = y/n,$$
$$\cos\theta = x/m.$$

Hence
$$\frac{x^2}{m^2}+\frac{y^2}{n^2}=1.$$

This equation is seen to represent an ellipse having O as its centre and m and n as its major and minor semi-axes.

From the position of the point O_{ac} (Fig. 81) it is evident that the centrodes of a relatively to c, and c relatively to a,

form a pair of circles of which the length of the link a is respectively the radius and the diameter. Hence it follows that the relative motion of a and c may be represented by the rolling together of circular curves of the same sizes as the centrodes in question—a point to which attention is again drawn (see § 57).

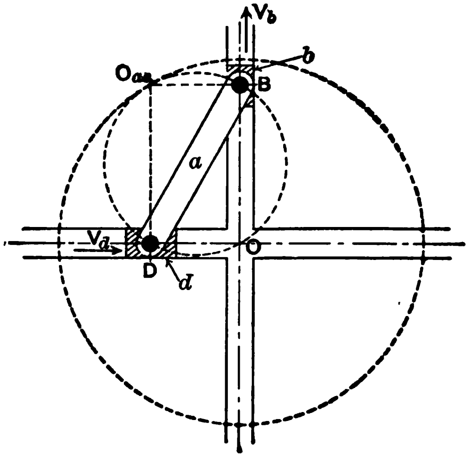


FIG. 81.

Notice that if V_d, V_b be the linear velocities of d and b respectively with regard to c, then, since $\omega_{ac} = \frac{V_d}{OB} = \frac{V_b}{OD}$, we have

$$\frac{V_d}{V_b} = \frac{OB}{OD}.$$

44. Oldham's Coupling.—The fourth inversion of the double slider-crank chain, when a is the fixed link, gives rise to a mechanism which has been used as a coupling for connecting shafts whose axes are parallel, and as an elliptic chuck, by means of which objects of elliptical cross-sections may be turned in an ordinary lathe.

Let Fig. 82 represent the chain when a is fixed. Evidently the locus of O is a circle of diameter BD.

Let O_1, O_2 be two positions of O; then, since O_1DO_2 and

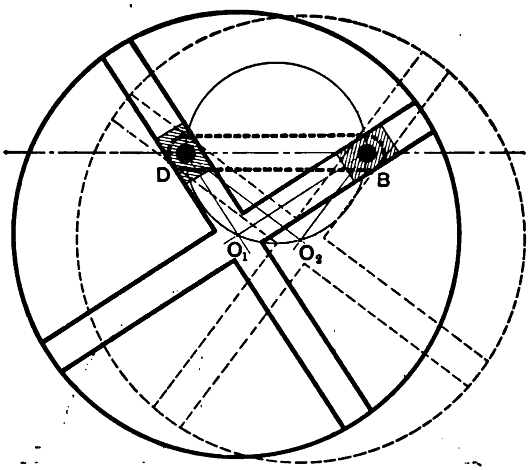


FIG. 82.

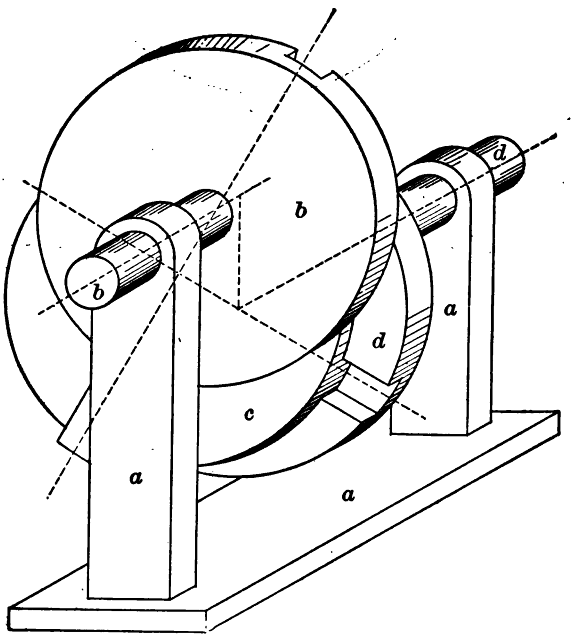


FIG. 83.

O_1BO_2 are angles in the same segment of a circle, they are equal; hence if b turns through any angle O_1BO_2, d turns through an equal angle O_1DO_2. By attaching a shaft to each of the links b and d we are thus enabled to communicate rotation from one to the other with uniform angular velocity ratio. Fig. 83 shows the form actually taken by

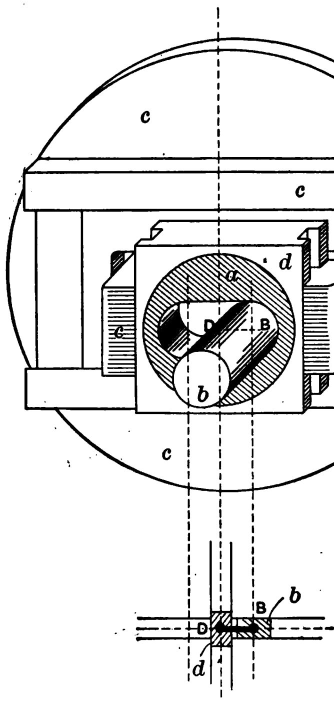


FIG. 84*a*.

this mechanism when used as *Oldham's coupling*. The link c becomes a disc having projecting feathers or keys on its faces, these keys being at right angles to one another and fitting into corresponding grooves on the enlarged ends of the two shafts b and d. The link a becomes a frame carrying the bearings of the two shafts.

Precisely the same kinematic chain is used in the case of the elliptic chuck, which was probably invented by Leonardo da Vinci.

Figs. 84*a* and 84*b* represent this contrivance, seen from the back, the face of the plate *c* to which the work is attached being turned away from view.

The plate *c* has behind it two straight pairs of guides at right angles to one another; the block *b* slides be-

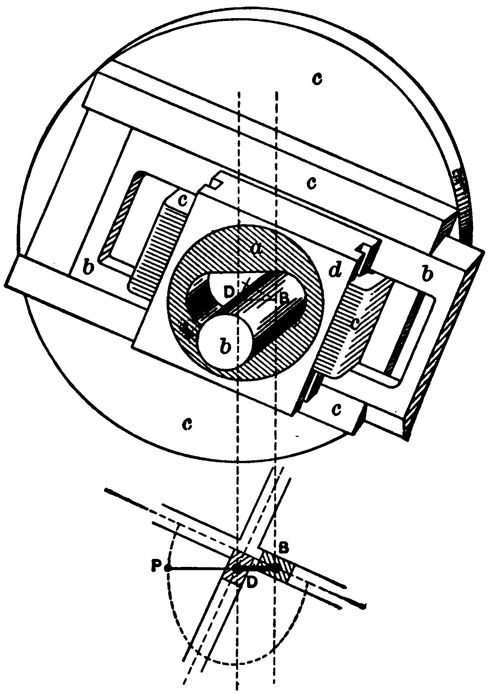


FIG. 84*b*.

tween one of these pairs of guides, while the block *d* moves between the other pair, which pass through slots cut for the purpose in *b*. The block *b* is secured rigidly by being screwed on to the nose of the lathe mandrel, with which it rotates. The mandrel passes through an oval hole in the eccentric *a*, which is clamped firmly (by screws which are not shown) to

the lathe headstock, in such a way that the distance BD between the axis of a and that of b can be varied as required. That distance is the effective length of the fixed link in the mechanism, and upon it depends the eccentricity of the ellipse to be described.

It will be seen that this construction corresponds exactly to the arrangement of Fig. 82. Accordingly it is evident that a point at rest with regard to the link a (as the point of a cutting-tool would be) will describe an ellipse on a piece of work attached to, and rotating with, the link c, just as a tracing-point attached to a (Fig. 80) was shown to describe an ellipse with respect to c in that case. It will be seen that the distance from the tracing-point P to D (Fig. 84*b*) is the semi-minor axis, while the length BD is the difference between the semi-axes.

It is obvious that a number of fresh mechanisms may be derived by changing the angle between the directions of motion of the two sliding pairs; in this case the chain would be known as a skew double slider-crank chain. Fig. 85 shows an example of such a chain, but space does not permit of the discussion of such mechanisms.

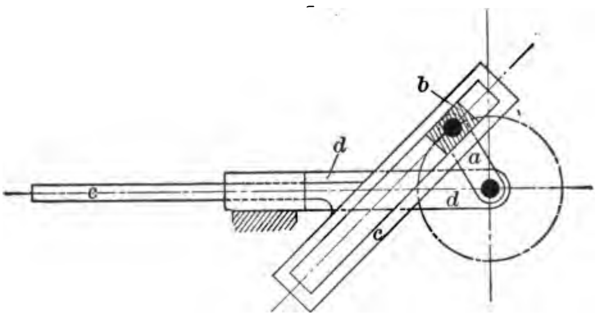


FIG. 85.

45. Crossed-slide Chains.—We proceed to consider the chain derived from the slider-crank chain by introducing a second sliding pair in such a way that each link contains one element of a sliding and one of a turning pair. As

distinguished from the chain just discussed (the double slider-crank chain), in which one link contains elements of each of two turning pairs, and another contains elements of each of two sliding pairs, we may call this the *crossed-slide chain*. It is essentially a crossed chain, just as the crossed slider-crank was, because the straight line in which the centre of one turning pair moves does not pass through the centre of the second turning pair. One of its forms is shown diagrammatically in Fig. 86, and is occasionally employed for

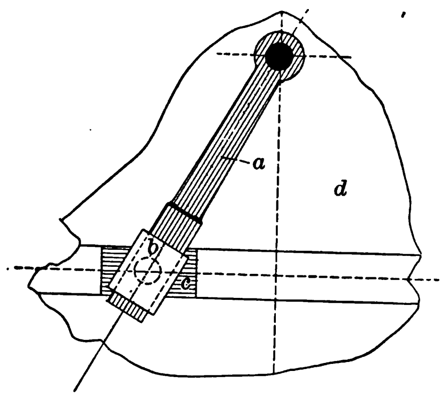


FIG. 86.

working the rudder of large ships, under the name of Rapson's slide. For this purpose it has the great advantage that the leverage increases as the helm is put over. In the figure the fixed link *d* represents the framework of the ship, *a* the tiller and rudder-head, and *b* is a block sliding on *a* and turning on *c*. The steering-gear moves the block *c* between guides secured to *d*, and thus actuates the rudder.

The same mechanism is employed for working the valves of duplex steam-pumps, in which each of the two steam-pistons works the valve of the neighboring cylinder, and it occurs again in a slightly different form in the arrangement of the compensating cylinders used in the Worthington high-duty pump for storing up the excess of energy exerted by

the steam during the first portion of the stroke of the piston, and restoring that energy during the later part (Fig. 87). Here d is the pump framework, a a are the compensating cylinders, rocking on trunnions attached to d; b b are the plungers which enter the cylinders against pressure during the first half of the stroke, and return during the later half; c is the pump-piston, rod, and plunger.

The virtual centres of the chain are shown in Fig. 88, and

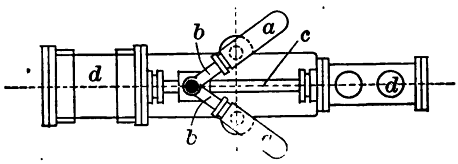


FIG. 87.

the pair of centrodes corresponding to the relative motion of b and d are shown in Fig. 89, construction lines being given for one point on each centrode.

Certain velocity ratios in this chain are of importance; for example, the ratio of the angular velocity of the tiller a to the linear velocity (V_c) of the block c relatively to its guides.

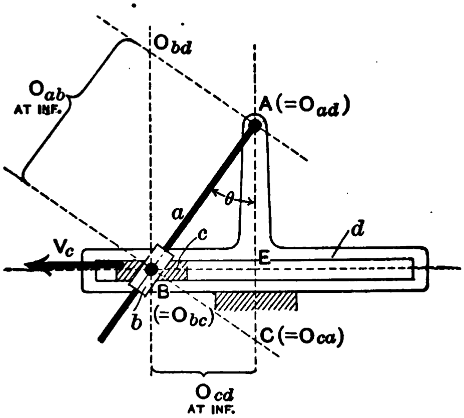


FIG. 88.

In Fig. 88 let the angle $BAC = \theta$; then, since O_{ac} is a point common for the instant to a and c, as a point in c it is moving

in a direction perpendicular to AC with velocity V_c. Its angular velocity (and therefore the angular velocity of a in which it is a point) about A is therefore $\frac{V_c}{AC}$, which is easily seen to be equal to

$$\frac{V_c}{AE}\cos^2\theta.$$

Hence if the block c has a uniform linear velocity, the angular velocity of the tiller varies as the square of the cosine of the

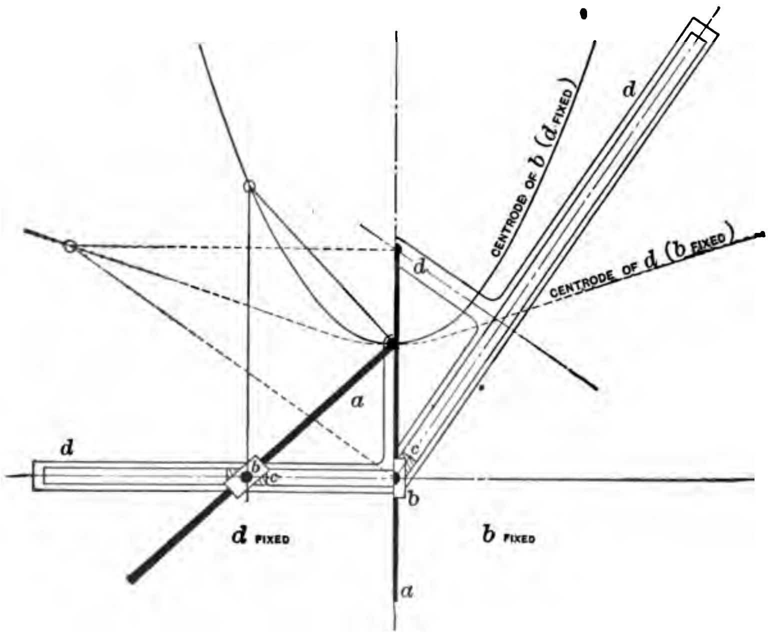


FIG. 89.

angle of helm. It is this property which gives the apparatus its value as a steering-gear; for it may readily be shown that if a constant force be applied to c, the turning moment on the rudder-head increases as the helm is put over; in fact the turning moment varies as $\frac{1}{\cos^2\theta}$.

It is easy to draw a curve of angular velocity for the link a. In Fig. 90 make $AE' = AE$ and draw BC perpendicular to AB. Draw EF parallel to CE', then $\frac{AF}{AE} = \frac{AE'}{AC}$;

therefore $\frac{AF}{AE^2} = \frac{1}{AC}$, and

$$\text{angular velocity of tiller} = AF \cdot \frac{V_c}{AE^2}.$$

Thus a series of points such as F will give us a polar diagram of the angular velocity of the tiller.

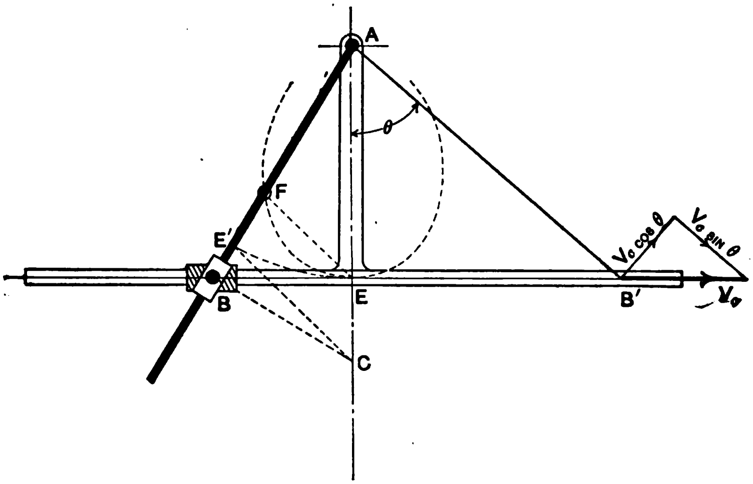


FIG. 90.

Notice that in any position we may look upon V_c, the linear velocity of the point B' along EB', as being the resultant of two velocities, $V_c \sin\theta$ along AB', and $V_c \cos\theta$ at right angles to AB'. The former gives the speed with which the block b is sliding along a; the latter shows that the angular velocity of a is

$$\frac{V_c \cos\theta}{AB'} = \frac{V_c \cos^2\theta}{AE},$$

the same result as that obtained previously.

Rapson's slide is only a particular case of the crossed-slide chain. It may be noticed, however, that we obtain the same mechanism whichever link is the fixed one, since each link has on it an element of a turning pair and also an element of a sliding pair.

46. Straight-line Motions Derived form Slider-crank Chain.—A number of straight-line motions have been devised which are really slider-crank chains. In such mechanisms the line described by the tracing-point is often only approximately straight, and when it is exactly so, its straightness depends upon the accuracy with which the flat surfaces of the sliding pair have been formed.

To this class belongs Scott Russell's straight-line motion, represented in Fig. 91. The link b in an ordinary slider-

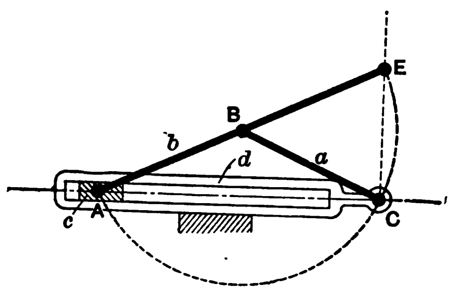


FIG. 91.

crank chain is extended to E, and $AB = BE = BC$. It is then evident that the angle ACE is the angle in a semicircle, and that the point E describes a straight line CE so long as A describes a straight line AC.

With other proportions of the lengths AB, BC, BE, approximate straight-line motions may be obtained. In Fig. 92, for example, suppose A and E to lie on the straight lines AC, EC, respectively; it has been seen that a point B will describe an ellipse (shown by the dotted curve), of which C is the centre, and AB and BE the lengths of the semi-axes. A circle may be drawn so as to cut this ellipse

in four points, as at P, Q, R, S, and if we connect B and F, the centre of the circle, by a rigid link, the path of the point E will cut the straight line CD in four places, supposing A traverses the straight line AC. By a suitable choice of the point F, the circular path of B may be made to differ

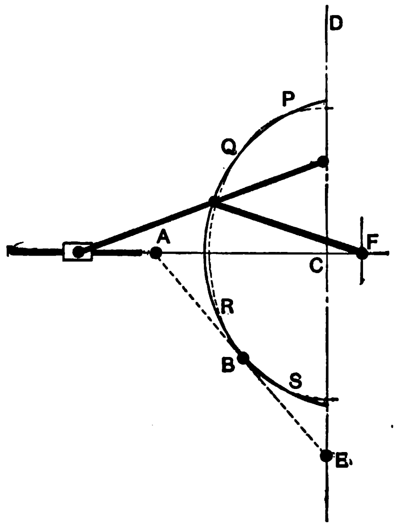


FIG. 92.

very little from the ellipse during a considerable range of movement, and the actual path of the tracing-point E will nearly coincide with the straight line CE.

In the second inversion of the slider-crank chain, in which b is the fixed link, a point on the link d may be chosen such that its path is approximately straight.

Thus in Fig. 93a suppose that a straight line AB, of fixed length, passes through a fixed point O, while a point C on it is compelled to traverse a straight line DE. The curves described by A and B are known as conchoids, and are shown by the dotted lines. It is possible in a swinging-block slider-crank chain to find a point P on the link d in such a position that while the circular path of O_{ad} coincides nearly with the

part BB' of the conchoid, and the centre of the pair bc cor-

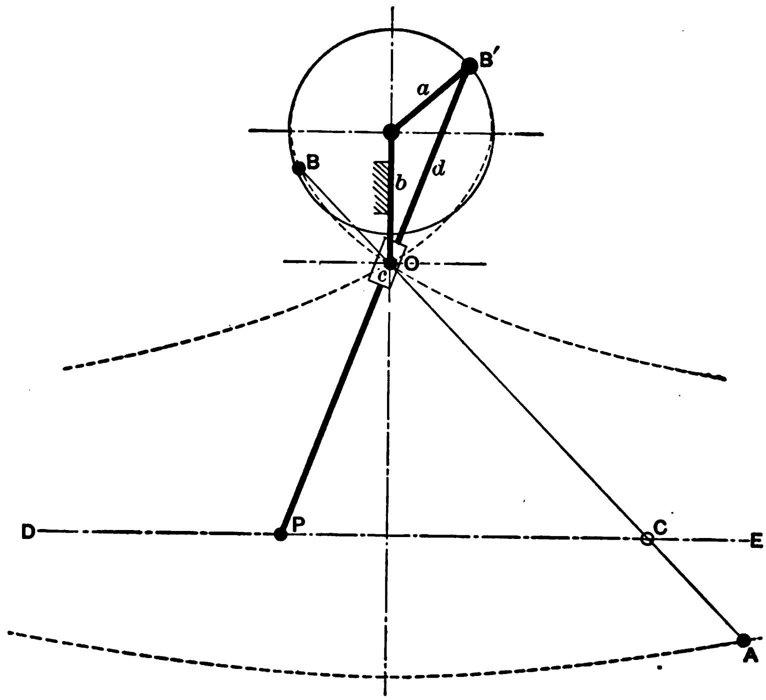


FIG. 93*a*.

responds with the point O, the path of the tracing-point P will nearly coincide with DE. Fig. 93*b* represents a model of such a mechanism, in which the point P is guided in an approximately straight line.*

47. **Chain Containing Sliding Pairs only.**—It is possible to construct a closed kinematic chain containing only sliding pairs. Such a chain consists of three links, a, b, and c. Of these b and c are blocks sliding on a common link a. A projection on b slides in a slot cut in c, thus completing the chain. If c is the fixed link, the chain will evidently be

* For discussion of a number of straight-line motions see Kennedy, Mechanics of Machinery, p. 417.

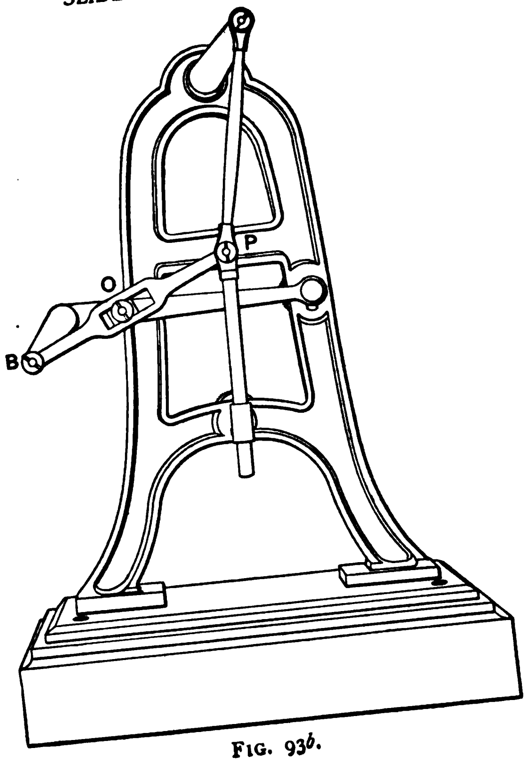


FIG. 93*b*.

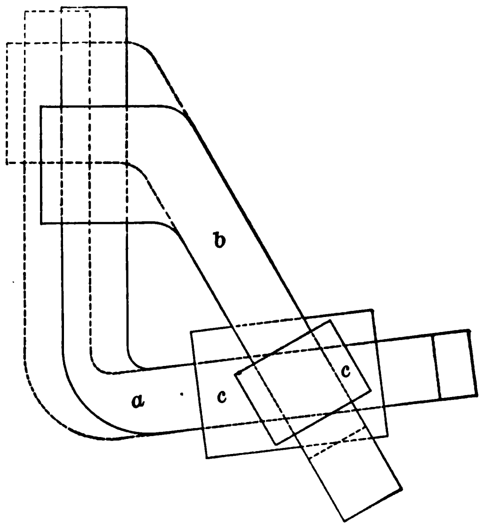


FIG. 94.

capable of being moved into such a position as that shown by the dotted lines in Fig. 94.

Such a chain exists, for instance, in most arrangements for adjusting bearings by means of wedges or cotters, as in the double-adjustment plummer-block (sketched diagrammatically in Fig. 95), in which the brass *c* has to be capable of slight movement in the direction of the arrow, to allow for wear, and is pushed forward by drawing down the wedge *b*. The pedestal itself and its cap form the link *a*, and the upward movement of the block or wedge *b* is prevented by some form of force- or chain-closure (see Chapter VI).

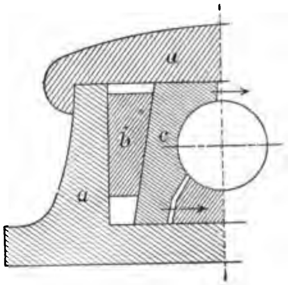


FIG. 95.

A chain containing four links and having four sliding pairs can also be devised, but, like the chain containing five turning pairs, it is not constrainedly closed.

CHAPTER V.

DETERMINATION OF VELOCITY AND ACCELERATION IN PLANE MECHANISMS.

48. Velocity and Acceleration Determined from Virtual Centres.—It is often necessary to determine the magnitude and direction of the velocity or acceleration of a given point of a given link in a plane mechanism. Such a calculation, for example, is frequently required if we wish to find the forces acting on a part of a machine when in motion, with a view to the correct proportioning of such a part to the work it has to do.

We have already studied this problem in certain cases, especially as regards the cross-head of a direct-acting steam-engine; the question has now to be discussed in a more general manner.

In a given mechanism, having given the velocity of a point on one link, and having found the positions of the various virtual centres, it is possible to determine the velocity of any point on any one of the links.

Take for example the beam-engine of Fig. 96, in which we suppose V_c, the velocity of the crank-pin, to be known. It is required to find the actual linear velocity (i.e., the velocity with relation to the frame or fixed link) of the piston and rod *b*.

Let *a* be the fixed link, *b* the piston, *d* the beam, *e* the connecting-rod, and *f* the crank.

First find O_{bd} at the intersection of a horizontal line through the beam centre O_{da} and the line joining O_{bc} and O_{cd}. Note that O_{ba} is at an infinite distance. Next find

O_{df}, draw a horizontal line through O_{af}, and find its intersection with the line joining O_{db} and O_{df}. This point is O_{bf}.

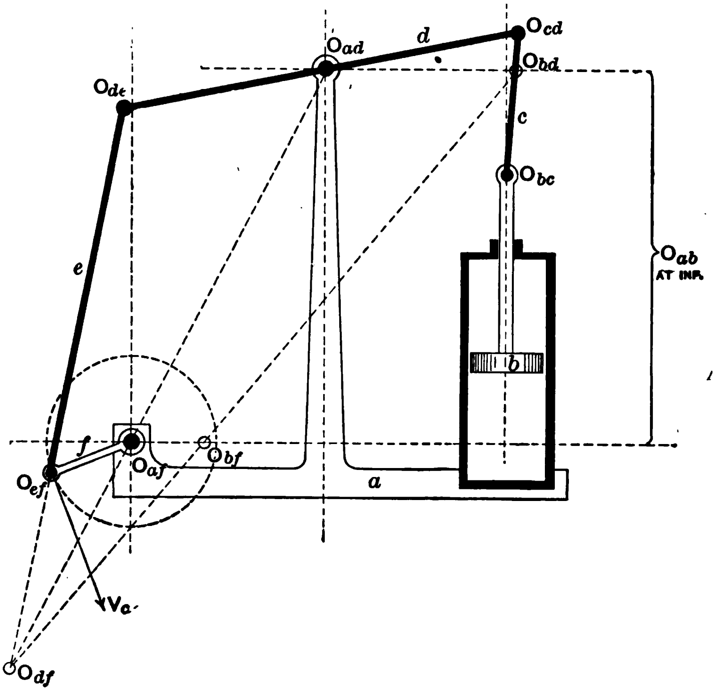


FIG. 96.

All these centres are readily found, remembering that they lie in threes in straight lines.

The point O_{bf} is a point common for the instant to the links b and f. Let the length of crank $=r$ and let the distance $O_{af} \ldots O_{bf} = m$.

Then the actual linear velocity of the point O_{bf} (considered as a point on the link f) must be $V_c \times \frac{m}{r}$, in a direction perpendicular to the line $O_{af}O_{bf}$, and this must also be the velocity of the link b, since O_{bf} is for the instant a point on that link also. The same construction will give the velocity of the piston for any position of the mechanism, except when the crank is on the dead-centre.

A similar method may be used in any case in which the various virtual centres can be found, but is not always possible for all positions of the mechanism, because many of the centres periodically recede to an infinite distance. This fact considerably reduces its practical usefulness.

49. Method by Using Point-paths.—The velocity of a given point on any link may be most simply determined for any given position of a mechanism by carefully drawing (to as large a scale as possible) the mechanism in two positions, one slightly before and the other slightly after the given position. The velocity of some one point of the mechanism being known, the velocity of the given point is readily found by comparing the displacements of the two points in the short time supposed to elapse between the two positions drawn, the direction of motion being known from the point-path on the drawing. It should be noted that this method of finding the velocity required is not susceptible of great accuracy, because the displacements whose ratio is measured must be supposed very small, in order that the result obtained may be as nearly as possible the true velocity of the point when the mechanism is actually in the given position. Hence the ratio of the displacements is difficult to measure. The method is nevertheless often used in practice.

As an example the mechanism of Fig. 97 may be taken. The figure shows Bremme's valve-gear.* It consists essentially of a lever-crank chain, the motion of the valve being taken from a point on one link produced. The figure, necessarily drawn here to a small scale, shows the proportions of an actual gear for a small marine engine. The eccentric of the engine corresponds to the crank of the lever-crank chain, and in practice coincides in angular position on the shaft with the engine-crank. The dimensions are:

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* See *Mechanical World*, September 2, 1889.

$AC = 1\frac{3}{4}''$ (throw of eccentric);
$CE = 10\frac{1}{2}''$;
$CD = 15\frac{3}{4}''$;
$EB = 14''$;
$AB = 20''$ (when the engine is going ahead).

The engine is reversed by altering the position of the suspension point B, as shown by the dotted arc.

It is required to determine the vertical component of the velocity of the point D (from which the valve is driven by a long rod) for any position of the gear, supposing the eccentric AC rotates uniformly at a speed of 170 revolutions per minute.

We first take a number of positions of AC (in this case 12 in one revolution), corresponding to equal small intervals of time (in this case 0.0294 second), and the corresponding positions of E and D are found. They are shown on the diagram and numbered successively, those of D forming points on a closed curve roughly oval in shape.

The vertical displacements of the point D have been plotted on a time base, giving the diagram xx. From this the velocity curve YYY has been drawn by the method of § 18. On determining the scale of the diagram we see that between the positions 4 and 5 the valve moved 0.60 inch upwards in $\frac{1}{12}$ revolution, i.e. in 0.0294 second. The velocity of the valve at the middle of that interval will therefore be approximately

$$\frac{0.60}{12 \times 0.0294} = 1.70 \text{ foot per second.}$$

In the same way the maximum downward velocity of the valve is found to be while the crank is moving from 10 to 11, and its value is about 2.00 feet per second.

If necessary the vertical *acceleration* of the valve can be determined as in § 19.

On drawing out the example for himself the reader will find that even if the mechanism be drawn full size great care is necessary to obtain anything like an accurate result.

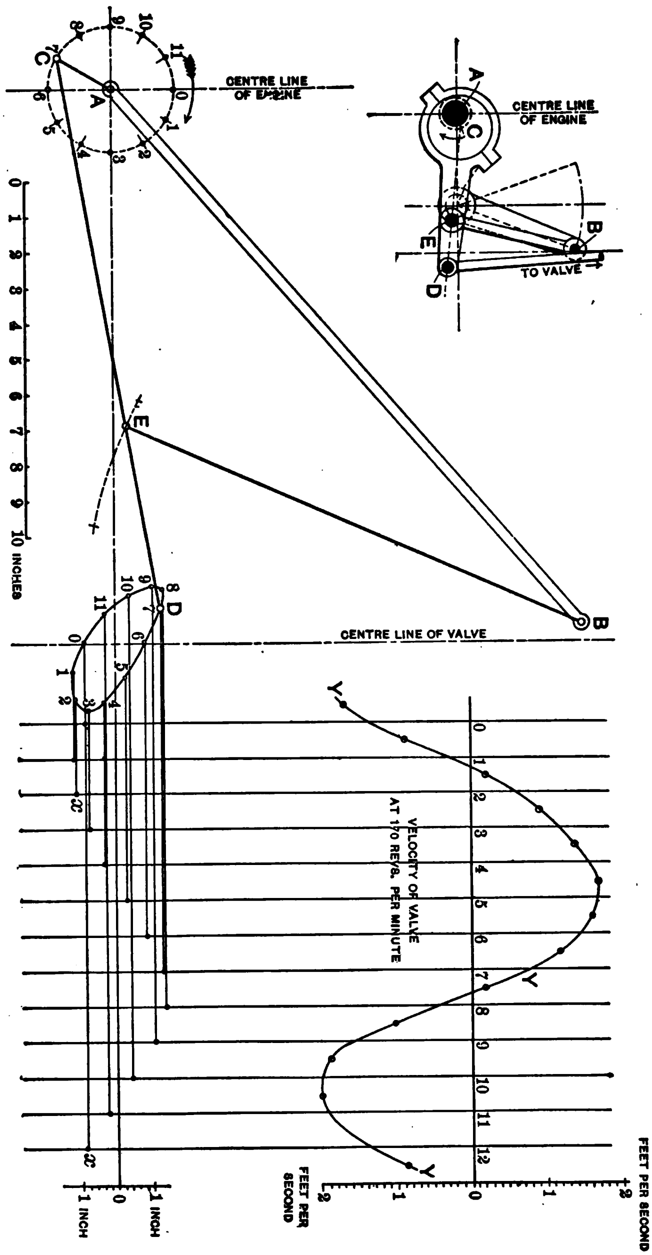

CENTRE LINE OF ENGINE
CENTRE LINE OF ENGINE
TO VALVE
10 INCHES
CENTRE LINE OF VALVE
VELOCITY OF VALVE
AT 170 REVS. PER MINUTE
FEET PER SECOND
FEET PER SECOND
1 INCH
1 INCH

FIG. 97.

50. Polar Diagrams of Velocities for Simple Plane Mechanisms.—The velocities in plane mechanisms can only be determined graphically from the positions of the virtual centres of the links when these centres fall within the limits of the drawing, and when their positions can be found with accuracy. Often the exact position of a virtual centre is difficult to define, because it lies at the intersection of two lines which make a very small angle with one another.

To avoid these difficulties, a general method of drawing diagrams for velocities and accelerations of points in mechanisms has been devised,* and a few simple cases will be considered here.

In Fig. 98 let ABC represent a rigid body having plane motion, and suppose the linear velocity v of the point A and the angular velocity ω of the whole body to be known. It is required to determine the linear velocities of the points B and C.

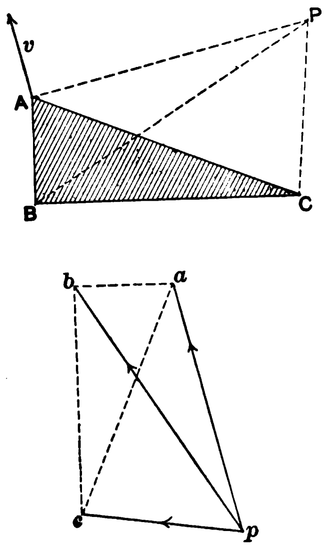


FIG. 98.

Let P be the virtual centre of ABC with regard to the plane of motion; then $AP = \frac{v}{\omega}$; hence the position of P can be found, since PA is perpendicular to the direction of v.

Join PB, PC. From these lines the directions of motion of B and C are known, and the magnitudes of the linear velocities are also known, since

$$\text{velocity of } B = \omega \times PB.$$

and

$$\text{velocity of } C = \omega \times PC.$$

* R. H. Smith, Graphics, Book I, Chap. IX ; Burmester, Kinematik, Chaps. XI and XII.

These velocities can, however, be determined (without finding the position of the virtual centre) as follows:

From any pole p draw the vector pa, representing the velocity v. From the point a draw $ab(=\omega \times AB)$ perpendicular to AB, and draw bc $(=\omega \times BC)$ perpendicular to BC. Then pb, pc are vectors representing respectively the linear velocities of the points B and C. The truth of this statement will be seen from the facts that the sides pa, ab are perpendicular to the sides PA, AB, and they are also proportional, since

$$pa = \omega \times PA,$$

and $$ab = \omega \times AB.$$

Hence $$pb = \omega \times PB \doteq \text{velocity of } B.$$

And similarly $$pc = \omega \times PC = \text{velocity of } C.$$

Note also that the triangles abc, ABC are similar; in fact abc is the *velocity image* of the body ABC, and is turned through an angle of 90° *in the same sense as that of the angular velocity* ω. The lines ab, bc, ca are of course vectors, and on consideration it will be evident that ab, for instance, represents the linear velocity of B (round A as centre), due to the actual angular velocity ω, because we have drawn $ab = \omega . AB$ and at right angles to AB. Further, the values of pb and pc have been obtained by vector-addition, the process described and explained in § 16.

Next suppose that we have to determine the velocities in a linkwork mechanism such as that of Fig. 99, where $ZPABC$ represents a chain of links connected by turning pairs, PZ being the fixed link.

The angular velocity ω of PA is supposed to be known, and also the directions in which the points B and C are moving at the instant considered

The point A is turning about the fixed point P; hence its direction of motion is at right angles to PA and its linear velocity is $\omega . PA = v_A$. We have to find v_B and v_C, the linear velocities of B and C.

Take any pole p, and draw lines pa, pb, pc from it par-

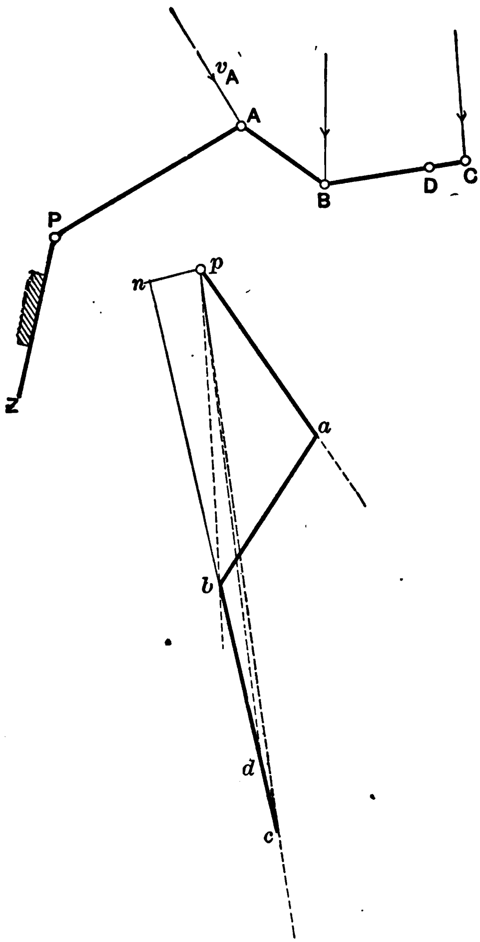


FIG. 99.

allel to the given directions of v_A, v_B, and v_C. Set off $pa = v_A$ and draw ab perpendicular to AB and bc perpendicular to

BC. Then pb and pc represent v_B and v_C on the same scale as that on which pa represents v_A.

The triangle pab, for example, is a vector triangle, or triangle of velocities, for the body AB, and ab is the velocity image of AB, just as in the previous case. The vector ab really represents the velocity of A with regard to B or of B with regard to A, according to the sense in which we measure it.

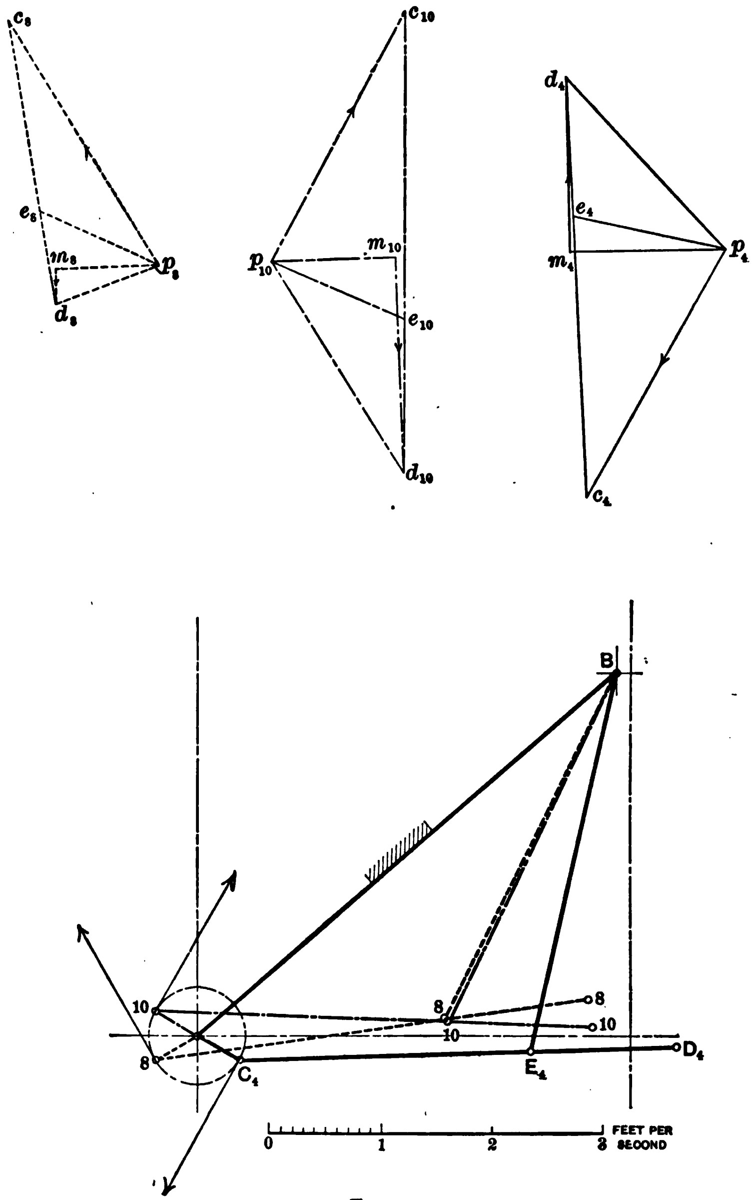
It is evident that the linear velocity of the point B will be due to two causes: (1) the velocity of A with regard to the fixed link ZP, and (2) that of B with regard to A. We also know that B can have no velocity along BA, for BA is a rigid body.

Hence to find pb (the velocity of the point B with regard to ZP) we compound pa (the velocity of A with regard to ZP) with ab (the velocity of B with regard to A). Similar reasoning holds good in the case of pc.

If cb be produced and a line, pn, drawn to cut it at right angles, it will be seen that pc is the resultant of pn (the velocity of C in the direction CB) and nc (the velocity of C in the direction normal to CB). Similarly nb is the velocity of B in the direction normal to CB, and hence bc is the velocity of C *with regard to* B, measured in a direction normal to CB, i.e., bc is the velocity of C about B. Note that since the link CB is rigid, C and B must have the same velocity, pn, along CB.

The diagram can be drawn in a similar way if some of the pairs are sliding pairs, and it will be found that if the chain of links has both ends attached to fixed points the velocity diagram becomes a closed polygon.

If it is required to find the velocity of a point D in the link BC, that of C being known, it is only necessary to make $bd = bc \times \frac{BD}{BC}$, and to join pd. The required velocity is then represented by pd. This is evident, since the veloc



c_8
e_8
m_8
p_8
d_8
c_{10}
m_{10}
p_{10}
e_{10}
d_{10}
d_4
e_4
m_4
p_4
c_4
B
10
8
10
8
10
C_4
E_4
D_4
0
1
2
3
FEET PER SECOND

FIG. 100.

ity of D about B is to that of C in the proportion of $BD : BC$, hence $\frac{bd}{bc}=\frac{BD}{BC}$. Thus bcd is seen to be the velocity image of BCD.

We may take as an example of the use of this construction the Bremme valve-gear of Fig. 97. In Fig. 100 the velocity diagrams have been drawn for the positions 4, 8, and 10 of Fig. 97. The diagrams have been drawn from separate poles for the sake of clearness, but they might equally well have been drawn from the same point as pole if that had been advisable.

Having drawn the mechanism in position 4 (say), and having found by calculation that v_c (the velocity of the centre of the eccentric) is 2.59 feet per second, a line p_4c_4 is drawn from the pole in a direction parallel to v_c, and of the proper length. We know that the direction in which E is moving is perpendicular to BE, and p_4e_4 is therefore drawn of indefinite length perpendicular to BE. The point e_4 is found by drawing c_4e_4 perpendicular to C_4E_4 and p_4e_4 then gives the magnitude of the velocity of E. To find that of D, we produce c_4e_4 to d_4, making $\frac{cd}{ce}=\frac{CD}{CE}$; then p_4d_4 represents the velocity of D when the mechanism is in position 4. The vertical velocity of the valve will be represented by d_4m_4, the vertical velocity of the point D to which it is attached, and on measurement this line is found to scale 1.56 feet per second. (Compare value shown by curve in Fig. 97.) In a similar way d_8m_8 and $d_{10}m_{10}$ are found, and so on for any required position of the gear.

It is not difficult to see that the velocity diagrams obtained by this method are really the same as some of those whose construction in certain special cases is explained in Chapters III and IV. For instance, Fig. 101 shows the construction already described for the piston velocity in a direct-acting engine, together with the polar method of

determining the same quantity. It is plain that the triangles BCE and bpa are similar and that pa and CE represent the same quantity to different scales. The vector pa thus represents the linear velocity of the piston, while pb

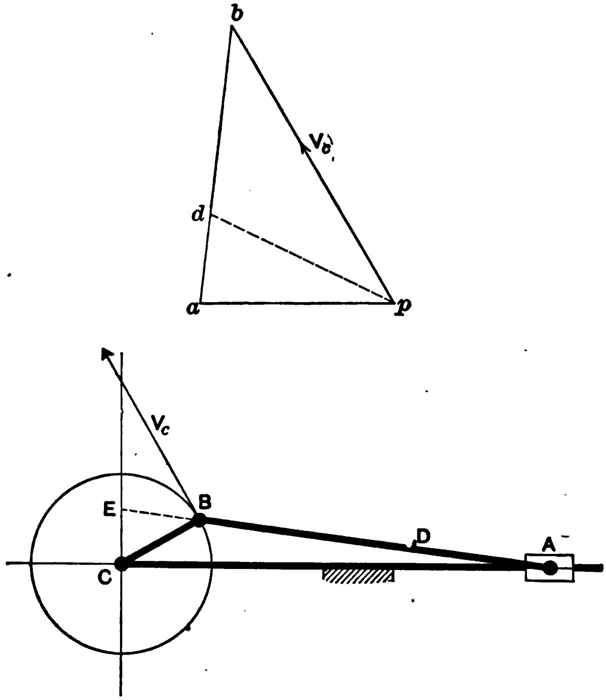


FIG. 101.

and ab represent respectively the linear velocity of the crank-pin B around C and that of the crank-pin B around A. ab is in fact the *velocity image* of AB, and pd gives the velocity of any point D on AB, if $\frac{ad}{ab} = \frac{AD}{AB}$. Since the triangles BCE and bpa are similar it also follows that we may look on BE as a velocity image of BA. It is, however, turned through an angle of 90° from the position ab.

51. Indirect Method in more Complex Cases.—It is not possible in every case to proceed in such a direct manner in

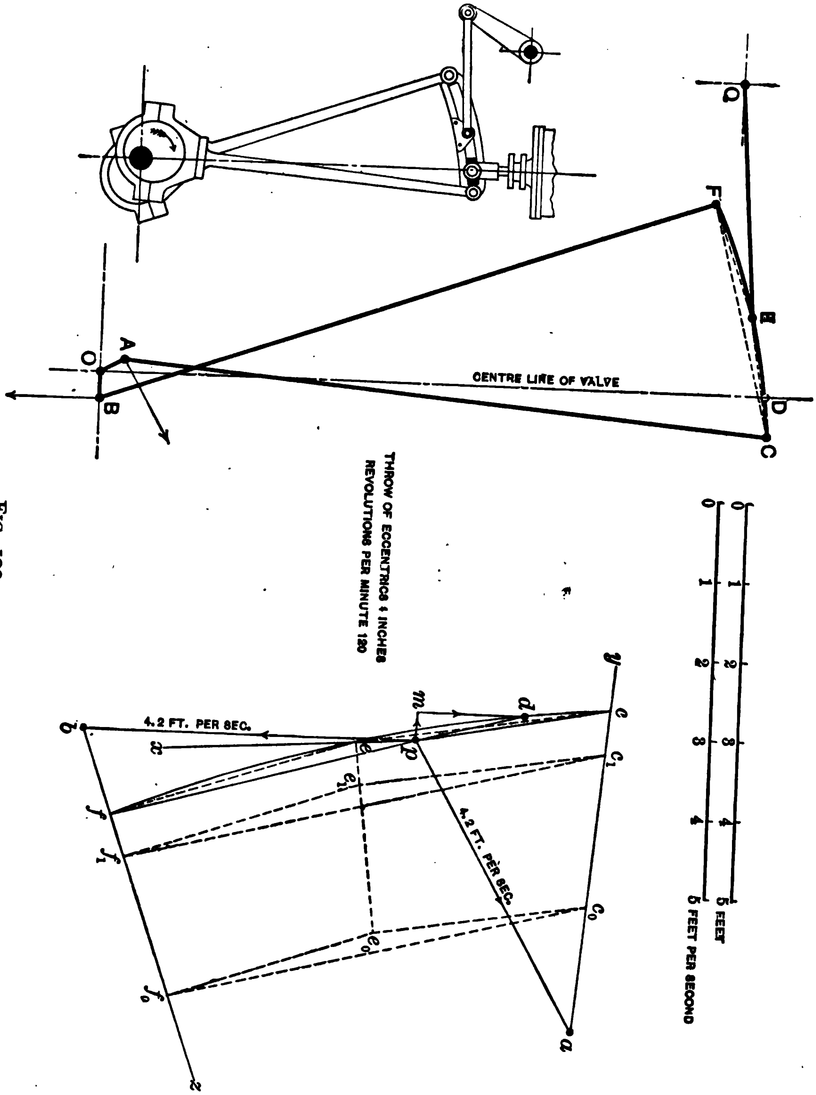


FIG. 102.

constructing the velocity diagram for a mechanism. Fig. 102 shows a link motion for working the slide-valve of a

steam-engine, of which OA, OB are the two eccentrics. These are rigidly connected and rotate uniformly about O, the centre of the crank-shaft; AC, BF are the eccentric-rods, CF the link, and QE is the drag-rod or suspension-link. Q remains fixed, except when the engine is reversed. We wish to find the velocity of a point D on the link CF. The motion for the valve is taken from the point D.

The actual velocities of points A and B are known, and also the direction of motion of E.

Having drawn out the mechanism in the required position, a pole, p, is taken and the vectors pa, pb drawn representing the linear velocities of A and B respectively. In the figure these correspond to an angular velocity of 120 revolutions per minute; they are each 4.2 feet per second.

A line px of indefinite length is next drawn at right angles to QE, and therefore parallel to the direction of motion of E. The point e must of course lie somewhere on this line. We next draw ay, bz of indefinite length, perpendicular to AC and BF respectively; the points c and f must lie somewhere on these lines.

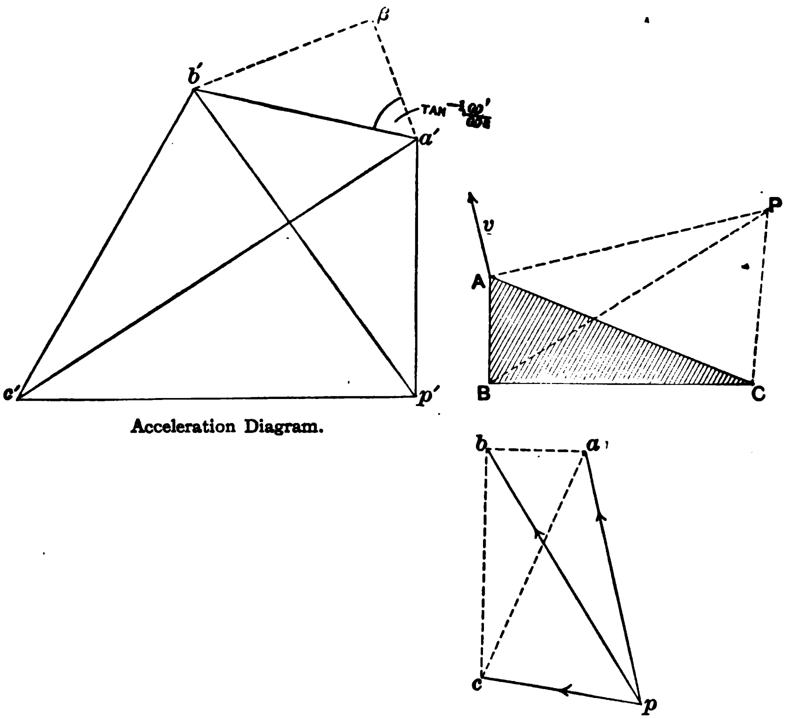
The required velocity images of the points C, E, and F must lie in some such position as $c_0e_0f_0$, where the triangle $c_0e_0f_0$ is similar to the triangle CEF, but is rotated through 90° in the sense of the motion. Another possible position would be $c_1e_1f_1$, and the line e_0e_1 will evidently pass through all possible positions of e. Thus e is found at the intersection of the lines e_0e_1 and px. We then draw ec and ef respectively perpendicular to EC and EF, and through the points c e and f draw a circular arc, which will be the velocity image of the curved link CEF.

The vectors pc, pe, pf then give the velocities of C, E, and F respectively. To find the velocity of D the point d is marked in its proper position on the curve cef, and pd is drawn. In the position shown this velocity is 1.42 feet per second, and is not in the direction of the centre line of the

valve-spindle. It is therefore necessary to attach the valve-spindle to a link-block capable of sliding on or in the link. The horizontal velocity of the point D and the actual vertical velocity of the valve will be represented by the lines pm and dm; they are respectively 0.35 and 1.36 feet per second.

The indirect method just explained has to be adopted to draw the velocity diagrams for many compound kinematic chains. The example here given will be sufficient to guide the reader in constructing such diagrams for most cases occurring in practice.

52. Polar Acceleration Diagrams for Plane Mechanisms. Acceleration Images.—Since accelerations, like velocities,



Acceleration Diagram.

Velocity Diagram.

FIG. 103

are vector quantities, it is plain that by similar constructions to those explained in the preceding section we can obtain

polar acceleration diagrams; there is one pole from which the vectors radiate, just as in the corresponding velocity diagrams.

Referring again to the rigid body of Fig. 98, suppose that we know the linear acceleration v' of the point A and also the angular velocity ω and angular acceleration ω' for the body. We wish to find the linear accelerations of B and C.

In Fig. 103 take a pole, p', and draw $p'a'$, representing to any convenient scale the known acceleration of the point A.

The acceleration of the point B may evidently be obtained by adding to the vector $p'a'$ the vector $a'b'$, representing the acceleration of B with regard to the point A.

Now, the acceleration of a point moving with *uniform* linear velocity v (or angular velocity ω) in a circular path is $\frac{v^2}{r}$, or $\omega^2 r$ radially (§ 15). But if the linear velocity be *not uniform* the point will be subject not only to the radial acceleration just mentioned, but also to a tangential acceleration directed along the path of the point and measured by $v' = \frac{dv}{dt}$, the rate of change of the magnitude of the velocity. This quantity may also be denoted by

$$\frac{dv}{dt} = \frac{d}{dt} r\omega = r\frac{d\omega}{dt} = r\omega',$$

where r is a constant. We thus see that in order to determine fully the acceleration of a point we must know not only the rate of change of *direction* of the velocity, or radial acceleration $(\omega^2 r)$, but also its rate of change of *magnitude*, or tangential acceleration $(r\omega')$, ω being the angular velocity and ω' the angular acceleration of the point.

The acceleration of B (supposing for the moment A to be fixed) will therefore depend on two components, namely, the known radial acceleration $\omega^2 \times AB$ along BA and the

known tangential acceleration $\omega' \times AB$ at right angles to AB. Compounding these two acceleration by vector addition, we see that the resultant will be $AB \cdot \sqrt{\omega^4 + \omega'^2} = a'b'$. Draw $a'\beta' = AB \times \omega^2$ and parallel to BA, and then $\beta'b' = \omega' \times AB$ at right angles to AB and in the proper sense given by the sign of the angular acceleration ω'. The vector $a'b'$ is thus found; it makes an angle $\tan^{-1}\frac{\omega'}{\omega^2}$ with AB. Note that the radial acceleration $a'\beta'$ is easily found from the velocity diagram, since its value is $\frac{(ab)^2}{AB}$.

The resultant of $p'a'$ and $a'b'$ is $p'b'$, which represents the acceleration of the point B. The line $a'b'$ is in fact the *acceleration image* of AB, and the acceleration of any point, C, on the moving body will be represented by $p'c'$, where the triangle $a'b'c'$ is similar to the triangle ABC.

Note that in these diagrams the acceleration image $a'b'c'$ is the original figure ABC altered in scale, and rotated through an angle $\left\{180^\circ - \tan^{-1}\frac{\omega'}{\omega^2}\right\}$ in the sense of ω. The pole p' in general does not correspond in position with the virtual centre, for p' represents that point of the body which undergoes *no acceleration* (not that point which has no velocity) at the instant considered.

In the case of the valve-gear of Fig. 97, whose crank is rotating with uniform angular velocity, the acceleration diagrams shown in Fig. 104 are constructed as follows, supposing the velocity diagrams to have been previously drawn. The acceleration of the point C is wholly radial, since ω is uniform; thus from the pole p' the vector $p'c'$ is drawn such that $p'c' = \frac{(pc)^2}{AC}$. Next, the direction and magnitude of the radial acceleration of E are known; the line $p'\epsilon'$ is therefore drawn parallel to EB, and of length equal to $\frac{(pe)^2}{EB}$.

The tangential component of the acceleration of E is at present only known as to direction; hence $\epsilon'x'$ is drawn of indefinite length and perpendicular to $p'\epsilon'$. The point e', which is the acceleration image of E, lies somewhere in this line.

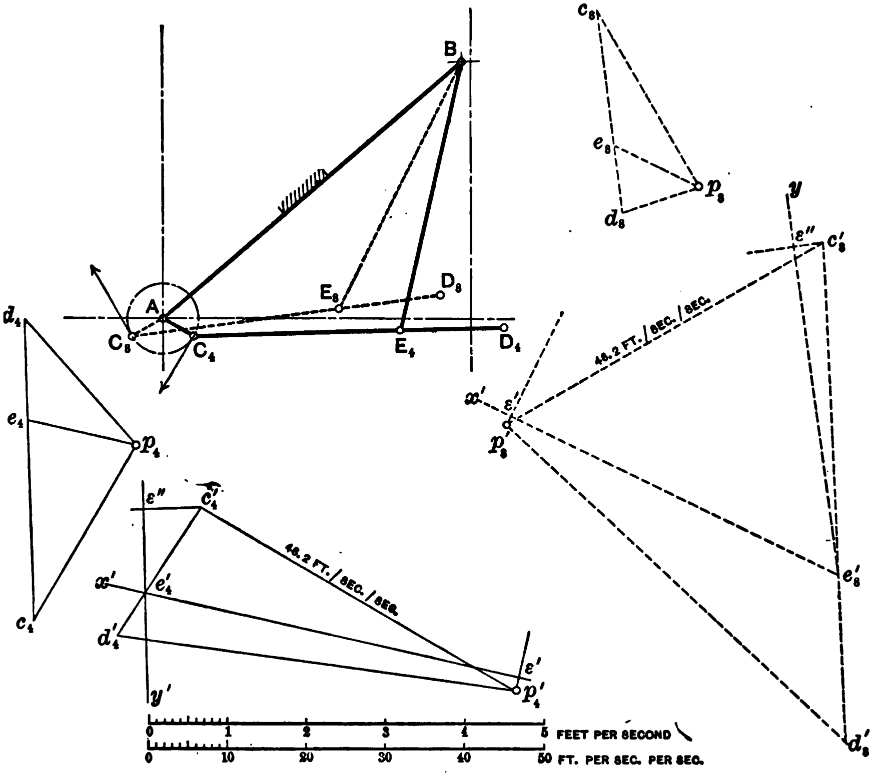


FIG. 104.

Now consider the acceleration of the point E with regard to the link AC, i.e., the acceleration of E round C. Its radial component is known; therefore draw $c'\epsilon''$ parallel to EC (*not* to CE) and of length $\dfrac{(ce)^2}{CE}$. The direction of the tangential component is known, hence $\epsilon''y'$ is drawn at right angles to $c'\epsilon'$, and the point e' must lie on this line. Thus

e' is found at the intersection of $\epsilon' x'$ and $\epsilon'' y'$, and $p'e'$ (not drawn in the figure) represents the actual acceleration of the point E. If $c'e'$ be produced to d', so as to be a proportional copy of CED, d' is the acceleration image of D, and $p'd'$ gives the acceleration of D.

In the examples drawn out in the figure, $pc = 2.59$ feet per second, while $AC = 1\frac{3}{4}$ inch $= 0.145$ foot. Hence the radial acceleration of C is $\dfrac{(2.59)^2}{0.145} = 46.2$ feet per second per second.

In the case of the slider-crank mechanism of the direct-acting steam-engine, the acceleration diagram is shown in Fig. 105.

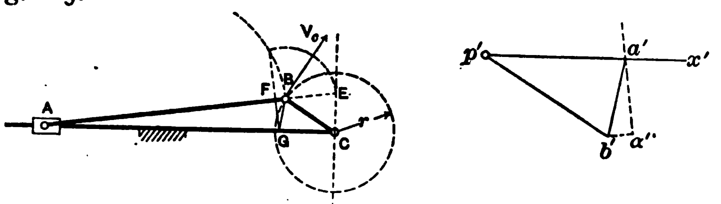


FIG. 105.

The vector $p'b'$ is first drawn, representing to a convenient scale $\dfrac{V_c^2}{r}$, the radial acceleration of the crank-pin B, and then $b'\alpha'$ is made equal to the radial acceleration of the point A around B; as found from the velocity diagram this quantity is equal to $\dfrac{(ab)^2}{AB}$ (see Fig. 101). The line $p'x'$ is drawn parallel to AC, and along it will be measured the acceleration of A, which has no component at right angles to AC. The line $\alpha' a'$ is drawn perpendicular to the direction of AB, and $\alpha' a'$ then represents the tangential component of the acceleration of A around B. The point a' is thus determined and the line $a'b'$ is the acceleration image of the connecting-rod AB.

It will be remembered (§ 36) that our construction for the acceleration of the piston was to set off along BA

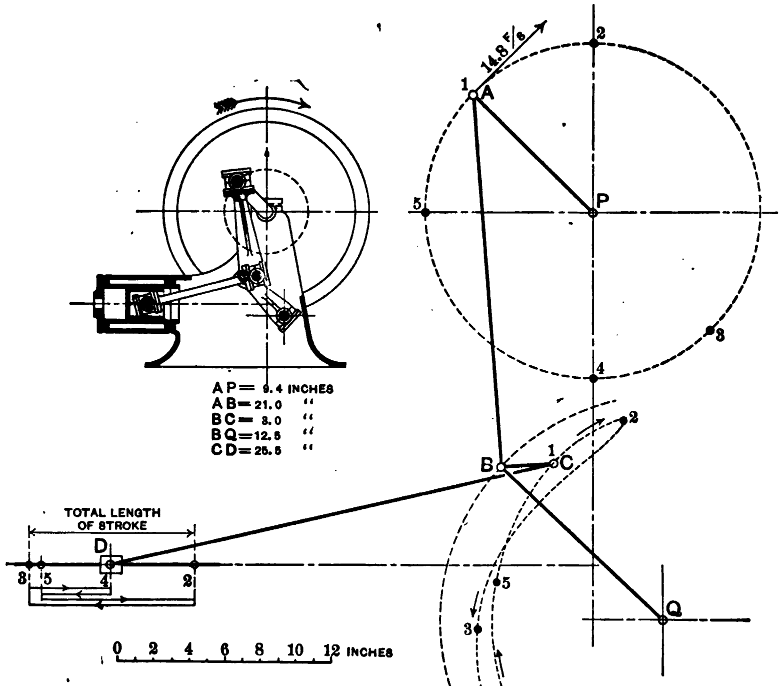
a length $BF = \frac{BE^2}{BA}$, and to draw FG perpendicular to AB, cutting AC in G. It has been pointed out that the line GB (see also Fig. 65) is actually an acceleration image of AB. This will be plain on comparing the triangles CBG and $p'b'a'$, which are easily shown to be similar, one being turned through 180° with reference to the other, so that GB and $a'b'$ are parallel.

53. Example of Polar Velocity and Acceleration Diagrams.—In the case of the Atkinson "Cycle" gas-engine, shown in Fig. 106, we have a good example of the use of polar velocity and acceleration diagrams in determining the velocity and acceleration of the piston in an engine of an unusual type.

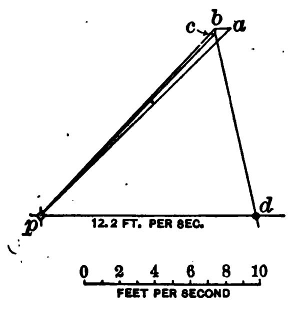
The essential feature of this engine is that for every revolution of the crank PA the piston (attached to the point D) makes two complete strokes of unequal lengths, its position being shown at points corresponding to the crank positions 1, 2, 3, 4, 5. This motion is obtained by connecting the piston to a point C on a link ABC, pairing with the crank-pin at A and with a rocking-lever, QB, at B.

Taking the mechanism in position 1, the linear velocity of the crank-pin at 180 revolutions per minute being 14.80 feet per second, we draw the vector pa, representing this velocity, and the lengths of pb and ab then give the magnitudes of the velocities of B around Q and of B around A, respectively, the directions of these velocities being, of course, known. The line ab is the velocity image of AB, hence the point c is easily found, remembering that the triangles abc and ABC are similar, but that one is turned through 90° with regard to the other.

We now know pc, the actual velocity of C with regard to the frame of the engine. The lines pd, cd are drawn respectively in the known directions of the velocities of D relatively to the frame, and of D about C. They intersect



VELOCITIES		ACCELERATIONS	
pa	14.8	$p'a'$	280
pb	14.2	$p'\beta_1$	194
pc	14.1	$a'\beta_2$	0.366
cd	10.5	$c'\delta'$	52.0
pd	12.2	$p'd'$	78



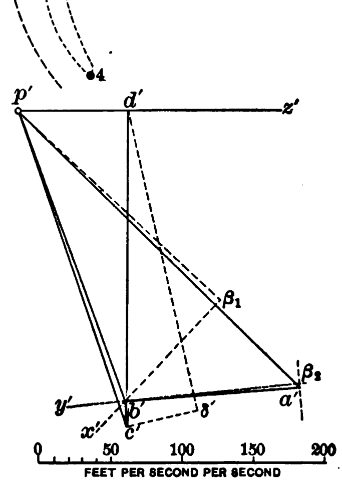


FIG. 106

at d, thus giving the magnitude pd of the velocity of the piston along its path. In the example drawn, the velocity of the piston is 12.2 feet per second, that of B is 14.2, and that of C 14.1 feet per second. The vector cd represents the velocity of D with regard to C.

We have now to draw the acceleration diagram. The acceleration of the point A is wholly radial, for the angular velocity of the crank is supposed to be uniform. We know therefore that this radial acceleration is $\frac{14.8 \times 14.8 \times 12}{9.4} = 280$ feet per second per second, and it acts in a direction parallel to AP. Again, we know from the velocity diagram that the velocity of B along its path is 14.2 feet per second; B therefore has an acceleration along its radius BQ of $\frac{14.2 \times 14.2 \times 12}{12.5}$ $= 194$ feet per second per second. Similarly the velocity of B about A is found from the diagram to be 0.8 feet per second as represented by the length ab, hence the acceleration of B along BA is $\frac{0.8 \times 0.8 \times 12}{21.0} = 0.366$ feet per second per second, and in a similar fashion the radial acceleration of D along DC is found to be $\frac{10.5 \times 10.5 \times 12}{25.5} = 52.0$ feet per second per second.

Starting from the pole p', the vector $p'a'$ is drawn parallel to AP and of length 280, measured to any convenient scale. The real acceleration of the point B with respect to A is not known, so that $a'b'$ cannot be drawn directly. We can, however, draw $a'\beta_2 = 0.366$, representing the radial component of the acceleration of B with regard to A. If $p'b'$ is the real acceleration of B, the point b' must lie somewhere on a line $\beta_2 y'$ drawn parallel to the direction of motion of B with regard to A, and passing through β_2, for $a'b'$ (the acceleration of B about A) must be a resultant of the radial acceleration $a'\beta_2$ and the tangential acceleration whose direction is $\beta_2 y'$.

In order to find another line on which b' must lie, we start again from p' and draw $p'\beta_1 = 194$, representing the radial component of B's acceleration about Q; a line $\beta_1 x'$ drawn through β_1 at right angles to $p'\beta_1$ and to BQ gives the direction of the tangential component, and it is therefore clear that b' is at the intersection of the lines $\beta_2 y'$ and $\beta_1 x'$. The vector $a'b'$ then gives the acceleration of B with respect to A, and is the acceleration image of AB. The point c', which is the image of C, is readily found by making the triangle $a'b'c'$ similar to the triangle ABC, and $p'c'$ gives the acceleration of C with regard to the fixed link or frame.

We may consider this acceleration as being the resultant of (1) the acceleration of D with regard to the frame, and (2) the acceleration of C with regard to D. We only know at present the direction of the first named, and can draw a line $p'z'$ parallel to the path of D. The radial acceleration of D about C is known to be 52 feet per second per second, and the vector $c'\delta'$ is drawn to represent this; note that $c'\delta'$ must be drawn parallel to DC and not to CD. Through δ' a line is drawn at right angles to DC and cutting $p'z'$ in d'. We have then $p'd'$ for the acceleration of D with respect to the frame, and $d'c'$ for the acceleration of C about D, while $d'\delta'$ gives the tangential component and $\delta'c'$ the radial component of this acceleration. Note that $c'd'$ is the acceleration image of CD. The acceleration diagram for any other position of the mechanism can be drawn by exactly the same method.

The foregoing examples will serve to indicate the system to be adopted in determining the velocity and acceleration of any point on a link of a rigid plane mechanism in any given position.

CHAPTER VI.

ALTERATION OF MECHANISMS. CLOSURE.

54. **Expansion of Elements.**—Certain examples have already shown the reader how widely the external forms of the links in a kinematic chain may be varied, while they still retain exactly the same relative motion (see Figs. 83 and 84, in which both mechanisms are the same inversion of the same chain).

We have now to consider further certain cases in which links of mechanisms are enlarged, reduced, changed in form, added, or omitted, without altering the relative movements of other links.

Perhaps the most familiar instance of a change in form, which in this case is really the *expansion of an element*, is to be found in the eccentric so generally employed for obtaining a reciprocating from a rotary movement in valve-gears and elsewhere, and shown in Fig. 97.

Let us suppose in a slider-crank chain that while the centres of the links remain the same, the radius of the cylindrical surface of the turning pair *ab* is increased, as in Fig. 107, until at length the crank becomes a disc, inside of which lies the centre of the pair *ad*.

The crank *a* has now taken the form of an eccentric, without in any way changing the relative motion of the links, the only alteration being that one element formed on each of the links *a* and *b* has been expanded.

Again, take the case already mentioned (in § 34) where in the quadric crank-chain a swinging link *c* has appar-

ently been replaced by a sliding block travelling in a curved slot, as shown in Fig. 60*b*. Notice that the pair *bc* remains just as before, while the appearance of the chain (but not the relative motion of its links) has been changed simply by

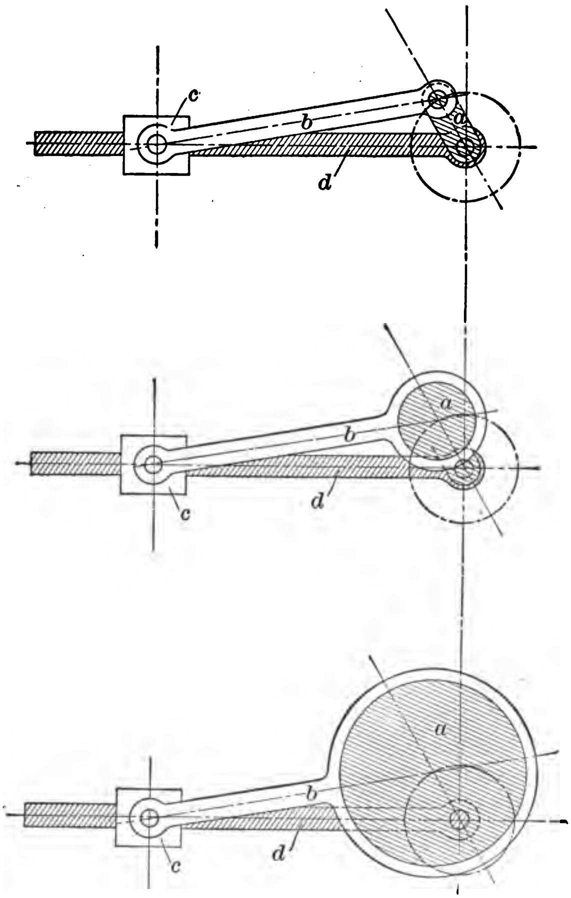


FIG. 107.

increasing the radius of the turning pair *cd*, and utilizing only a portion of its curved surface. The effective or kinematic length of the link *c* remains unaltered.

As a third example, imagine the radius of the pair *bc*,

in the slider-crank chain of Fig. 108, to be increased as shown, until it is greater than the length of the link *b*, while the link *a* retains its original form, that of a crank. The expansion may be carried a stage farther, as shown in Fig. 109, by increasing the radius of the pair *ab* also, but in a

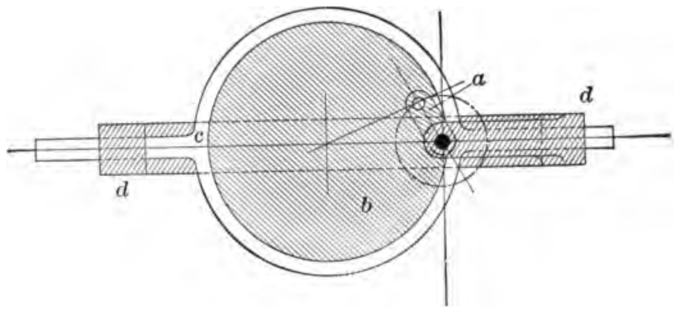


Fig. 108.

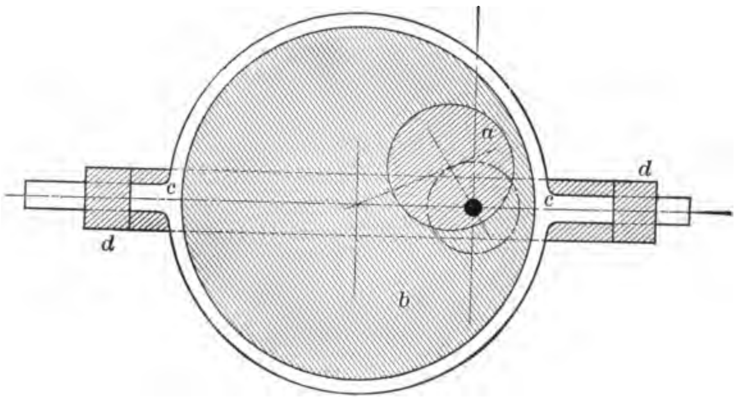


Fig. 109.

lesser degree. The links *a* and *b* have now both become eccentrics, while *c* takes the form of a strap provided with projections sliding in guides formed on *d*. The kinematic lengths, however, of the links in Fig. 108 are just the same as in Fig. 109, and the relative movements are the same.

Here we have instances of the expansion of the pairs of elements *bc* and *ab*.

55. **Augmentation of Chains.**—Many instances occur in which typical kinematic chains are apparently disguised by the introduction of additional links. Such a change is called by Reuleaux the *augmentation* of a chain or of a mechanism, and the links which are added, while giving the chain no new kinematic properties as a whole, are introduced for constructive reasons.

Take as an instance a bicycle wheel and its axle. Here the movement of the wheel relatively to the frame to which the axle is attached is exactly the same as if the connection between them were a simple turning pair. But on examination we find that the actual pairing is of quite a different character, and that a series of balls running in grooves cut in the hub of the wheel, and in the axle, have been provided, to minimize friction and wear.*

Some examples occur in which a relative motion that might have been attained by simple pairing is arrived at by the use of a whole chain, and such cases might equally well be looked upon as instances of augmentation.

A complex train of toothed wheels is often employed to give a velocity ratio which might have been obtained by a chain much simpler mechanically, but occupying more space, or inadmissible for some other reason. Again, in a steam-engine indicator the piston and rod are guided, not by a simple sliding pair, but by a straight-line motion, which is in itself a kinematic chain. Many other instances might be cited in which pairing is replaced by "chaining," that is, by the introduction of linkage; the link or links introduced being so arranged as not to alter any of the relative motions already existing.

It has been pointed out by Reuleaux that a chain which already possesses the largest number of links which it can

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* See Fig. 225.

have as a simple closed chain, necessarily becomes a compound chain on augmentation.

We shall see later that chains have occasionally to be thus augmented for purposes of closure.

56. Reduction of Chains.—A *reduced mechanism* is obtained by the omission of one or more of its links, corresponding alterations being made in the pairing of the remainder. To illustrate this we may take the gear used for actuating the valves of duplex non-rotative steam-pumps, which has been already mentioned and is shown in Fig. 110.

This mechanism is essentially a reduced crossed-slide chain (a crossed-slide chain is shown in Fig. 86, § 45).

Using the same letters as in that figure, we see that the link *b* has been omitted, the lever *a* has its lower end formed into a figure of constant breadth measured in the direction of motion of *c*, and that end is fitted easily between two parallel lugs or projections on *c*. It is plain that the relative movement of *c* and *a* is practically unchanged.*

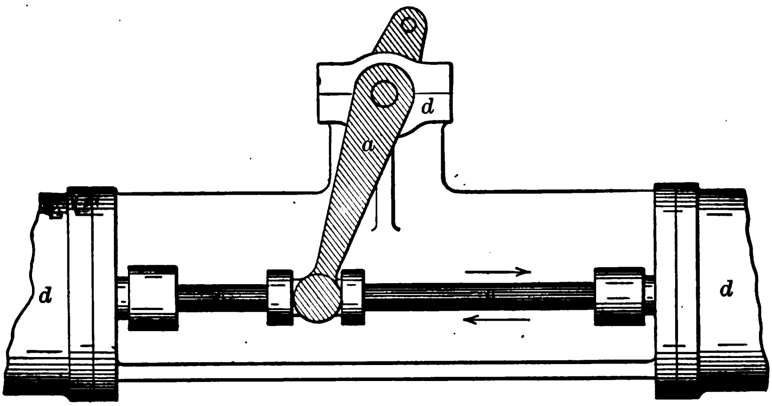

FIG. 110.

We have here, as a result of omitting the link *b*, the introduction of *higher pairing* with its advantages and disadvan-

* This statement is correct only for small displacements of the link *a* from its mid-position.

tages, among the latter being the mechanical trouble which will arise when the material of the links wears and the fit of the lever end becomes slack. The reader will find it easy to discover for himself numberless similar examples of augmentation and reduction in machinery which he has the opportunity of examining.

57. Reduction by Use of Centrodes.—Interesting cases of reduction occur sometimes in which chains are reduced by the introduction of higher pairing between new links taking the form of centrodes of links of the original mechanism. This can only be completely carried out where the centrodes take the form of closed curves. Consider, for instance, the form of the quadric-crank chain in which opposite links are equal, while the longer links cross one another. This chain and its virtual centres are shown in Fig. 111*a*. It is evident from symmetry that the sum of the distances from O_{ab} and O_{ad} to O_{ac} must be constant, and equal to the length of one of the longer links. Hence, if a or c is the fixed link, the path of O_{ac} will be an ellipse of which the length of the link a or c is the focal distance. The two ellipses will always touch at the point (O_{ac}) which describes them, and, in accordance with the well-known property of centrodes, may be imagined to roll on one another as the links move. Now suppose that d is the fixed link, and imagine that elliptical plates in the form of the pair of centrodes are attached to a and c (Fig. 111*b*). On removing the link b, we then get a mechanism (Fig. 111*c*) of three links only, and if proper constraint were applied * the links a and c would roll on one another, at the same time having exactly the same angular velocity ratio as if directly connected by a link b. We have thus reduced the mechanism and introduced higher pairing without affecting the relative motions of the remaining links.

* See §§ 3 and 4.

The employment of links shaped thus in the form of centrodes must not be confused with the method of transformed or reduced centrodes adopted by Reuleaux for the

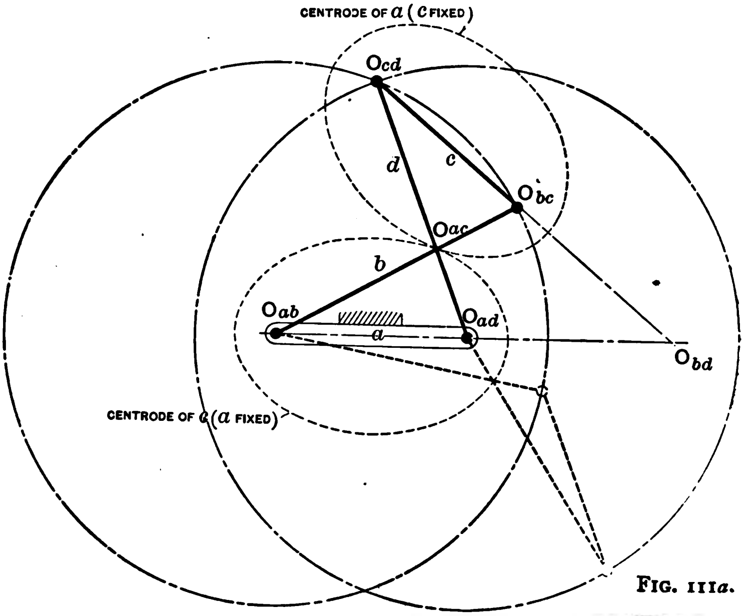

FIG. 111*a*.

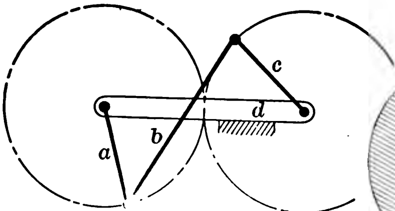

FIG. 111*b*.

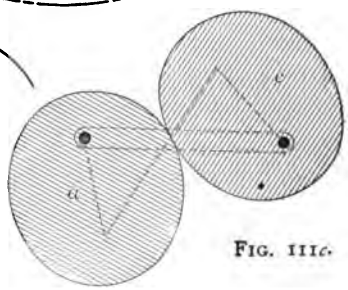

FIG. 111*c*.

purpose of expressing the velocity ratio of certain links in the simpler mechanisms.*

It is worthy of notice that the second pair of centrodes

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* Reuleaux, Kinematics, p. 70.

in the anti-parallel crank mechanism (those of the two longer links) are a pair of hyperbolas. The reader should draw these as an exercise.

58. Closure of Incomplete Pairs.—The meaning of the term closure as applied to pairs and to kinematic chains was explained in §§ 3 and 4. We have now to discuss the methods of applying such closure or restraint in various cases in which the pairs or chains would otherwise be incomplete.

On examination it is found that in a large number of pairs of elements existing in actual machines the forms of the elements are not such as to completely constrain their relative motion. For example, in certain forms of axle-boxes for cars or locomotives we find that the brass of the bearing embraces only a comparatively small angle on the upper surface of the journal, and hence the form of the bearing does not render separation of the surfaces in contact impossible. Such separation does not occur in practice, for the reason that the weight of the car presses the brass against the journal. We have here an example of *force-closure* of a pair. It is often necessary and convenient to employ force-closed pairs, since they are so readily taken apart for examination, and are usually simple in form. The table or platform of a weighing machine, for instance, generally rests on its knife-edges without being held down in any other way than by gravity. The force of gravity is not the only one employed for closure of pairs; in friction-gearing, for example, the rollers must be pressed together by some external force, so that one wheel can drive the other without slipping. We shall find also that force-closure has very frequently to be applied in pairs involving non-rigid links.

Although, strictly speaking, all pairs in mechanisms must be closed, either by the forms of their surfaces or by the application of an external force, cases occur in which

the desired object is attained by making the pair of elements into a kinematic chain complete in itself, in which the added links have for their only object the provision of the necessary restraint. This *chain closure* is more generally applied for the purpose of constraining the motion of incomplete chains, as will be presently seen.

59. Closure of Incomplete Chains.—As in the case of a pair, we may say that a kinematic chain is incomplete if the relative movements of its parts are not completely defined.

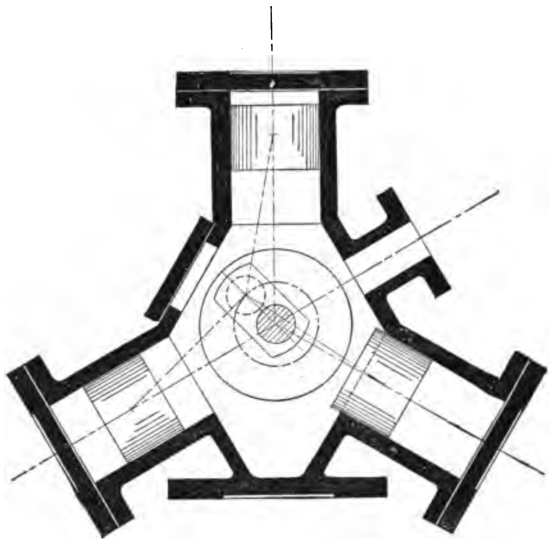

FIG. 112.

A chain may be incomplete (*a*) because it contains too many links, or (*b*) because it has not enough pairs of elements, or (*c*) either dead-points or change-points (§ 30) occur at which it is locked, or its motion becomes indeterminate.

Closure may be applied in an incomplete chain (1) by a force, (2) by the duplication of the mechanism, (3) by the addition of a pair of elements or another link.

When the incompleteness of a chain consists solely in the incompleteness of a certain pair, this may be rectified

by any of the methods just discussed. We need only consider therefore how to treat chains consisting entirely of closed pairs, the motion of which is indeterminate on account of the existence of dead-points or of change-points.

The flywheel of a single-cylinder steam-engine is an example of the use of the first method, *force-closure*, in passing dead-points, for it is the energy already stored up in the flywheel which keeps the crank rotating in positions where the steam pressure is not able to exert any turning moment on the shaft.

The energy of a moving body, such as a flywheel, is evidently not available when the machine to which it is attached is only moving very slowly or is just about to start. In these cases the second method, *chain-closure*, must be employed. For example, in a locomotive, or in a marine steam-engine, the mechanism is so arranged that

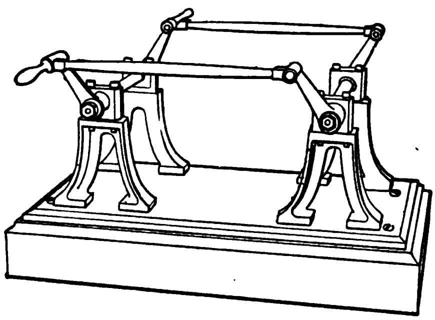

FIG. 113.

when one crank is at its dead-point, another, actuated by a separate steam-cylinder, piston, and connecting-rod, is in a more advantageous position. The original chain has thus been closed by *duplication of the mechanism.*

Another instance may be taken. In the well-known three-cylinder or Brotherhood type of single-acting steam- or hydraulic-engine (Fig. 112), three complete sets of driving apparatus work on the same crank, with the result that only one set can at any instant be passing its dead-point.

The parallel-crank mechanism of Fig. 113 is an instance of closure by the employment of a fifth link, which is adopted to enable the chain to move past a position which is not only a dead-point, but also a change-point. It has been already mentioned in § 30 that the necessary closure is obtained in this mechanism, when used on locomotives, by the addi-

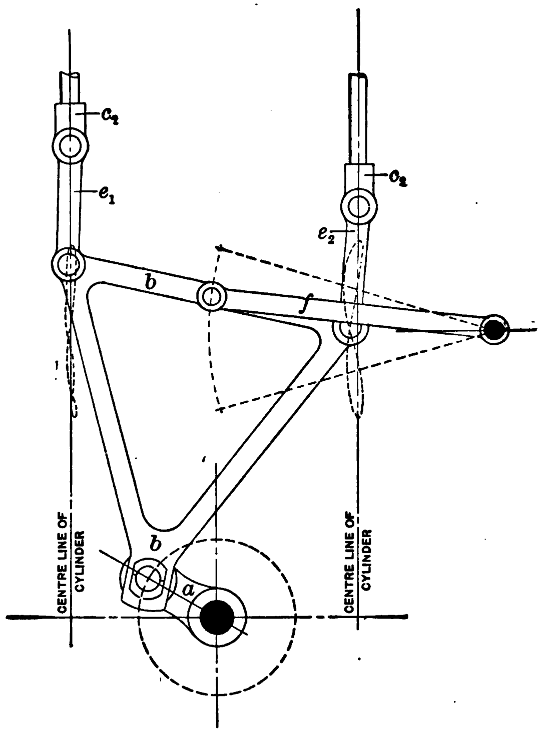

FIG. 114.

tion of another pair of cranks in another plane and a second coupling-rod. Fig. 113 represents a model which illustrates this arrangement. The same object may be attained by the placing of a third crank in the same plane as the first two, when the coupling rod takes the form of a triangular plate; thus a fifth link has been added. Fig. 114 shows

the link-work of a form of steam-engine * in which dead-points are avoided by the addition of new links and the partial duplication of the chain. Here the connecting-rod b is transformed into a triangular frame, and is paired (1) with the crank a, (2) with a link, e_1, connecting with the crosshead c_1, (3) with a link, e_2, attached to a second crosshead c_2, and (4) with a guiding link, f, attached to a fixed point on the frame. It will be seen that when the piston which actuates c_1 is unable to exert any turning moment on the shaft, the piston of c_2 is very nearly in the most favorable position with regard to the crank. In this way an engine is obtained which has no dead-point, and which has a very uniform turning moment compared to that of the ordinary single-crank engine. The fixed link or framework of the engine, d, is not shown in the diagram.

On consideration, it is obvious that in a chain which has change-points, but is otherwise closed, we can obtain complete closure if pairs of elements are arranged corresponding in form to the required motion, and coming into action or constraining the motion of the chain at the required instants. Such *pair-closure* of chains is, of course, the converse of the *chain-closure* of pairs, which has been already discussed. We shall see in a later chapter how to form a pair of elements, in general, so that they may have any desired relative plane motion, and shall find that most frequently such elements will have higher pairing.

As an example of pair-closure of a chain, one method of closing the anti-parallel crank chain at its change-point is shown in Fig. 115.

A pin P and a gab G are placed on the links b and d respectively, so that they are in contact at the proper instant, and permit only one kind of motion at the change-point. To do this the pin P must be placed on b at the point where it is cut by the centrode of d with regard to b,

* See *Engineering*, October 28, 1892.

and the gab must be where d is cut by the centrode of b with regard to d.

The pin and gab may be considered as portions of the two centrodes, applied for the purpose of closure as in Fig. 111, and so shaped as to prevent any relative slipping. In a similar manner the links a and b might be paired by using as pins and gabs portions of their elliptical cen-

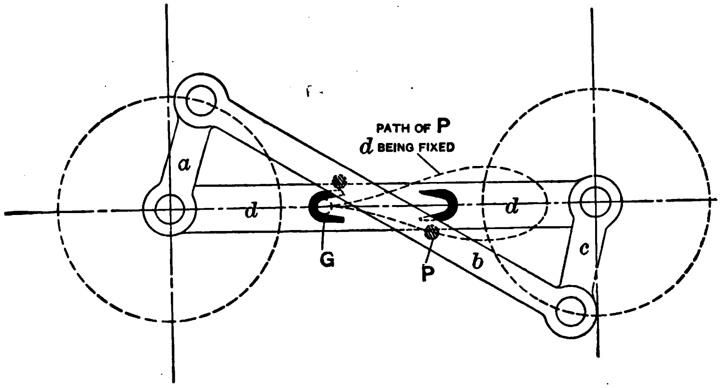

FIG 115.

trodes. Such centrodes would be equivalent to small portions of elliptical wheels, toothed to prevent slipping.

Many instances occur in which pairs of elements are thus applied so as to act at the change-points of a mechanism, and prevent its transformation into another mechanism or into a pair of elements. In general, pair-closure must be provided for *each* change-point; in the mechanism of Fig. 115 the *two* pins and *two* gabs are required so as to come into action at the *two* change-points occurring in each complete revolution of the chain.

CHAPTER VII.

CONSTRAINT AND VELOCITY RATIO IN HIGHER PAIRING INVOLVING PLANE MOTION.

60. Constraint of Bodies having Plane Motion.—It has already been stated that a body free to move in a plane possesses three degrees of freedom and has three degrees of constraint. Further constraint may be applied by causing such a body to touch certain points on the surface of a second rigid fixed body, these points being known as points of restraint. A *point of restraint* of a figure or body may be defined as a point on its outline, so touched by a point on the outline of a second fixed figure or body, that no relative sliding motion is possible along or parallel to the common normal to the two figures at the point of contact. When thus restrained the body or figure is considered as being kept in contact with the point or points of restraint.

We may take an example to illustrate the meaning of this definition, and to show the actual nature of points of restraint. Suppose (in Fig. 116*a*) that it is required to arrange a support or base, *a*, for a tripod, *b*, so that an instrument fixed on *b* can be removed from its support and replaced exactly in its previous position. This may be effected by providing *b* with three rounded points or legs, *CDE*. A hole, *F*, is made in the base, *a*, and is of pyramidal or conical form, so that if the rounded end of *C* is placed in *F*, there will be contact at three points of restraint; in this way, so long as the contact is maintained, the only possible relative motion of *c* and *a* will be one of rotation about some axis

passing through the centre of the spherical surface of the end of C. The next step is to provide on a a slot or groove, G, of triangular cross-section as shown; when D is placed in this groove there will be two more points of restraint, and the only possible relative motion remaining will be a rotation about the axis CD. Finally the position of b is fixed relatively to a if the third point E is made to rest in contact with a flat surface, H, formed on or connected with a, thus furnishing the sixth point of restraint required (see § 7).

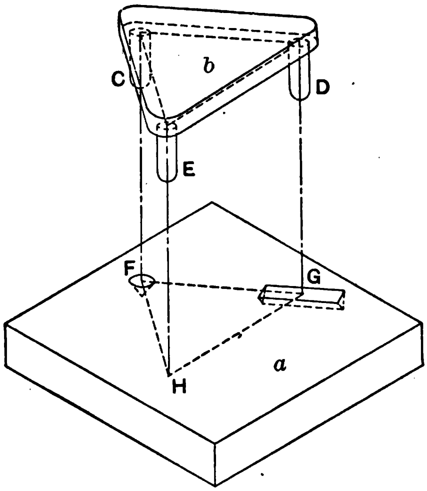

FIG. 116*a*.

The whole device is known as the "hole, slot, and plane."

The application of similar principles is illustrated in the design of Ewing's extensometer,* an instrument for measuring the deformation of test-pieces under stress. In this apparatus the bar or test-piece whose extension or compression is to be measured (a in Fig. 116b) carries a clip, b, attached by the points of two set-screws in such a way that b can move relatively to a about the axis of the set-screws at B. The clip b carries a projection, b', ending in a rounded

* Ewing, Strength of Materials, p. 75.

point F. This point engages with a pyramidal or conical hole formed on a second clip, c, which is also secured to a by means of two set-screws at C. So long as F rests in its recess, b and c can have no relative motion unless the length BC alters; in that case the angular motion of b and c will be proportional to the extension or compression of a. Actually the projection b' is not rigidly attached to b, but can turn

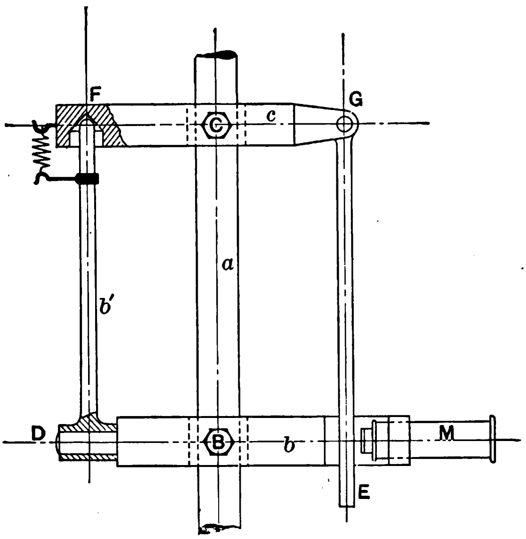

FIG. 116*b*.

through a small angle about the axis BD. This provision is made in order that any minute twist of the test-piece a about its axis BC may not affect the angular motion of b and c to any appreciable extent. This angular movement is indicated by the scale E attached to c; the distances CF, CG are equal, so that the movement of the scale, as read by the microscope at M, will be twice the actual deformation of the test-piece as taken on the length BC.

Similar methods are followed in designing the so-called kinematic clamps and kinematic slides.*

A *kinematic clamp* is a contrivance intended to fix completely the position of one body with reference to another; a *kinematic slide* permits one body to have one degree of freedom with reference to another.

On consideration it is plain that in a kinematic clamp or slide the points of restraint must be suitably placed with regard to the shape of the body to be restrained.

It is thus proper to inquire what must be the disposition of the points of restraint required, either to define the position of one body relatively to another, or to permit the movable body to retain one degree of freedom, and thus to constrain its motion completely. We shall suppose that the movable body at first possesses three degrees of freedom, and is capable of plane motion.

Let a (Fig. 117) be such a body, and let a fourth point

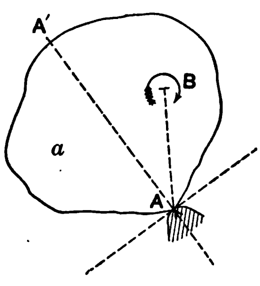

FIG. 117.

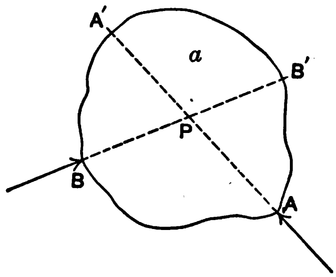

FIG. 118.

of restraint, A, be provided, in addition to the three points necessary for insuring plane motion. The arrow-head then represents the fourth point of contact of the restraining or fixed body.

Draw AA' normal to the tangent of the outline of a at A.

Any possible motion of a may be regarded as an instan-

*For an example of a kinematic slide, see Min. Proc. Inst C. E., Vol. CXXXII, p 49.

taneous rotation about a virtual axis perpendicular to the plane of motion. We need therefore only consider how the single point of restraint, A, affects the possibility of turning the body a about such an axis. It must be remembered that by the definition of a point of restraint, a is to be kept in contact with the restraining body. This is impossible if the virtual centre is not somewhere along AA', for if the virtual centre were, say, at B, a point about which only right-handed rotation is possible, it is plain that such rotation could only occur if the point A ceased to touch the restraining body. Hence we see that any possible instantaneous motion of a must be about a virtual centre situated in AA', and any motion of translation must be along a line at right angles to AA'.

Next consider the effect of keeping the body a in contact with a restraining body at two points, A and B. Let the normals AA', BB' intersect at P. The body is then only capable of an instantaneous rotation about P. If the normals are parallel, then only an instantaneous motion of translation, i.e., rotation about an infinitely distant axis, will be possible.

On adding another point of restraint, C (Fig. 119), it will be found that if we suppose that the body a remains in contact with the three new points of restraint, A, B, C, no movement is possible, except when the three normals intersect in one point or are parallel. In these cases instantaneous turning about the point of intersection, and instantaneous translation about an infinitely distant axis, are respectively possible, so that a at the instant considered will thus possess one degree of freedom and will have constrained motion.

In Fig. 119 a little consideration shows that no movement at all is possible except about an axis situated within the triangle PQR, so long as the restraining body is rigid. The whole field of motion, with the exception of PQR, then

becomes what Reuleaux calls a "field of restraint." But if movement did occur about an axis placed within the triangle PQR (in the figure such rotation could only be right-handed), the body a would at once cease to touch the restraining points with which we suppose it to be kept in contact. A similar result will be found with other arrangements of the points of restraint, and therefore in general

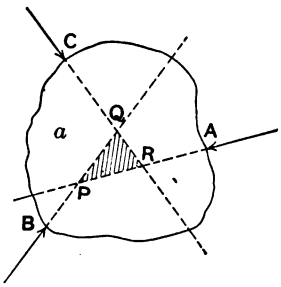

FIG. 119.

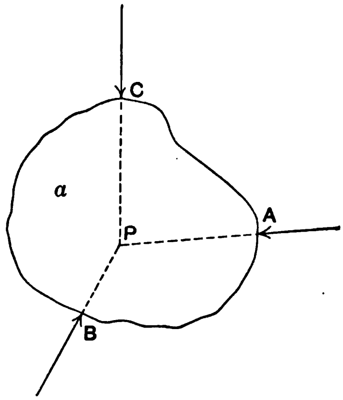

FIG. 120.

the position of a will be fixed if it is made to touch the restraining body at the three additional points A, B, C, a result already stated in § 7.

Fig. 120 shows the case in which the three normals meet in a point, P. If the shape of the body a is such that no point of restraint can be so applied as to have a normal that does not pass through P, then the body cannot be fixed by the application of three or any number of points of restraint, and its shape must be altered for that purpose. For example, a circular disc having plane motion could not be so fixed.

It is thus evident (a) that if one of two rigid bodies capable of relative plane motion remains in continuous contact with two points of restraint formed on the second body, the relative motion is constrained, and the virtual centre of the two bodies is always at the intersection of the two common normals.

Also (*b*) if three points of restraint are employed, and contact at all three is continuous, constrained relative motion is only possible if the three common normals intersect in a point, or are parallel.

The reader will find that the constraint of the motion of a body by means of such points of restraint as have been defined above is an easier matter than the limitation of the movement of a body by points of contact with a second fixed body, if no force is supposed to keep the two bodies in contact. In this case the bodies would possess greater freedom of motion than under the restrictions we have supposed. The theory of constraint has been treated by Reuleaux * and by Burmester,† to whose works the student is referred for information on the subject.

61. Closed Higher Pairs having Plane Motion.—Let us next suppose that the moving body *a* and the second or fixed body *b*, while kept in continuous contact, have such forms that one is the geometrical envelope of the other, and that in every position the normals at the several points of contact are either parallel or meet in a point. It is obvious that in this case at any instant *a* can move in one way, and in one way only, with reference to *b*; in other words, *a* and *b* will form a closed pair. We proceed to consider some examples of such pairing, which in general will be higher pairing, in accordance with the definitions in §. 2.

In Fig. 121*a*, let *ABCD* be a figure (called by Reuleaux a Duangle), drawn by describing the arcs *ABC*, *CDA*, with a radius equal to *BD*, and with *D* and *B* as centres respectively. Suppose that this figure, representing a body, *a*, having plane motion, is made to touch two lines, *PQ*, *QR*, inclined at an angle of 60°, the points *E* and *F* on these lines forming points of restraint for the duangle, and the lines *PQ*, *QR* representing the profile of the restraining body *b*. The normals at *E* and *F* to *QR* and *QP* will inter-

* Reuleaux, Kinematics, Chapter III. † Burmester, Kinematik, Chapter V.

sect at O, where they make an angle $FOE = 120°$, and they must pass respectively through the points B and D, since these points are the centres of the arcs ADC, ABC.

As the duangle moves in contact with PQ and QR, the

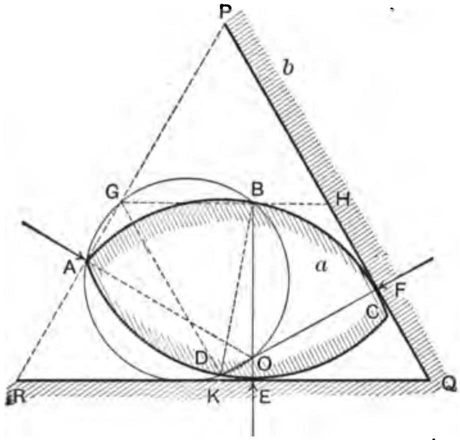

FIG. 121*a*.

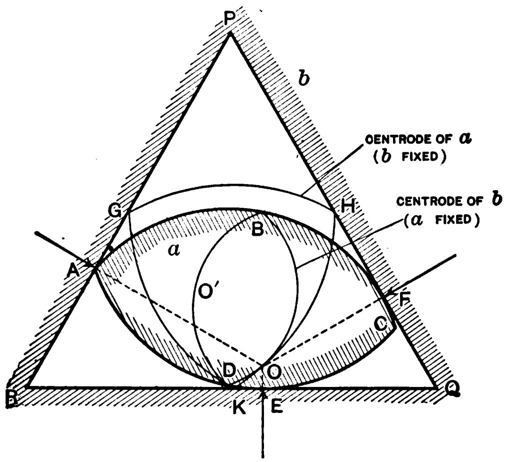

FIG. 121*b*.

path of B must be a straight line, GBH, parallel to QR and at a distance, BE, from it. The path of D similarly must be a line, GDK, parallel to QP. Hence the motion of the

duangle relatively to PQ, QR will be the same as that of a straight line of constant length, BD, whose ends lie continually upon two lines, KG, HG, enclosing an angle of 60°; further, the virtual centre of the two bodies will be the point O, the intersection of the two common normals.

Since the angles OBG, ODG are right angles, a circle may be drawn on GO as diameter, passing through the points GB, OD. The point A also lies on this circle, since the angle BAD is 60°. Join AG. Then so long as the curves ABC, ADC touch the lines QP, QR respectively, the angle $AGD =$ angle $ABD =$constant. Thus A lies continually on a line, RP, drawn through G and inclined at 60° to RQ. PQR is then an equilateral triangle, inside of which the duangle moves. The relative motion of the triangle and the duangle will be constrained if AO is the normal to PR at A; i.e., if the three normals at the points of contact meet at O. This is seen to be the case, for the angles AOD, ABD, ADB, AOB are all equal. Hence AO bisects the angle BOD and is perpendicular to PR.

The path described by O with reference to the triangle PQR is the centrode of the duangle. It evidently consists of a curve joining K and H. Now in any position the circle drawn on GO as diameter and passing through B and D has a chord, BD, of constant length, and the angle BGD is constant. Hence GO, the diameter of this circle, is the same (length $=GH$) for all positions of O. Thus O lies on a circular arc joining K and H and having G as centre, and the complete locus of O with regard to the triangle is an equilateral curve-triangle GKH (Fig 121*b*). Since the angle BOD is constant, the locus of O with regard to the duangle is seen to be a duangle $BODO'$, the radius of whose sides is $\frac{1}{2}GO$.

The whole relative motion of the duangle $ABCD$ and the triangle PQR is thus represented by the rolling of the duangle $BODO'$ inside the curve-triangle $GHOK$. The centrode of $ABCD$ with regard to the triangle PQR is $GHOK$;

that of the triangle with reference to *ABCD* being *BODO'*. Any point on the duangle *ABCD* will have a path made up of trochoidal curves described on the plane of the triangle *PQR*, and *vice versa*.

Relative motion of the duangle and the equilateral triangle may evidently be represented by the rolling together of a pair of circular arcs, one having a radius twice that of the other. Points on either figure will therefore describe trochoidal curves on the other.

The example just given will indicate the method of studying the relative motions of the elements of higher pairs having plane motion. A large number of closed higher pairs may be devised by utilizing figures of constant breadth. The equilateral curve-triangle previously mentioned is such a figure, and its motion relatively to a circumscribed square may be followed as an exercise.

A number of other forms are given by Reuleaux in the chapter already quoted. The student should note in all these cases that the form of the path described on *b* by a point on *a* is not the same as that described on *a* by the corresponding point on *b*, a condition previously mentioned as being characteristic of higher pairing.

62. Form of Elements for a Given Motion.—Having illustrated the method of determining the centrodes and the relative motion in the case of higher pairs of mutually restraining elements of given profile, we have next to show how to solve the converse of this problem, namely, how to find the forms of a pair of elements whose relative motion is previously decided. The relative motion in question must, of course, be defined by the forms of a pair of given centrodes, the mutual rolling of which, as already stated, represents the relative motion required. It most frequently happens in practice that we have also given the form or profile of one element of the pair, and the form of the second has to be found.

Let AA and BB (Fig. 122) be a pair of centrodes, of which A belongs to, or is traced upon, a body whose profile is aa'. It is required to find the profile of a second body to

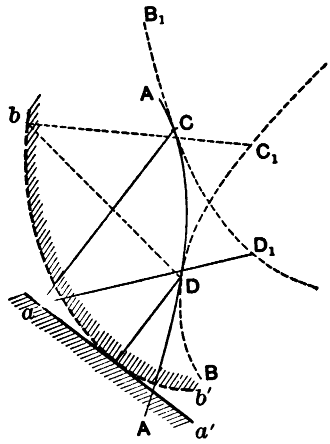

FIG. 122.

which the centrode BB belongs; the profile to be such that while the two bodies remain in continuous contact the centrodes will roll on one another and the bodies will thus have the desired relative motion.

Take any point, a, in the profile aa' and draw aC normal to aa' at a, and cutting the centrode A at C. In this case for convenience aa' is shown in the figure as a straight line, but it may, of course, be of any form.

At the instant when the profile bb' (to be found) touches the given profile at the point a, aC must be the common normal, and the virtual centre of the two bodies must lie on this normal, for otherwise contact would not be continuous. The point C, where the normal at a cuts the centrode A, must at that instant be the virtual centre of bb' with regard to aa', since the curve AC is the locus of the virtual centre of b. AC may be regarded as being attached to aa', since it is a curve traced on the body represented in outline by aa'. We proceed to find a point, b, on the profile of the second or

moving body, such that when a and b are in contact C is the virtual centre of the two bodies and aC the common normal.

Suppose that the centrodes AA and BB are in contact at some point D, and measure along the centrode BB a length DC_1 equal to the length of DC measured along AA. Draw B_1CD_1, representing the centrode B in the position it occupies when a is the point of contact of the two bodies and C is their virtual centre, and make $CD_1 = CD$. Join aD_1.

Then since the outline of bb' may be regarded as attached to the centrode B, any point on that outline having the same position in relation to C_1 and D that the point a has in relation to C and D_1 will be the point that touches a when the centrodes touch at C. Accordingly we need only make $bD = aD_1$ and $bC_1 = aC$ in order to determine the position of b. The point b is then a point on the required profile which will touch the point a when C is the virtual centre of the two bodies. In the same way we can determine any other point on the profile required, and it only remains to provide the resulting body with the restraint required to prevent any other motion than that desired. This would in general be done by so forming the body bb' that it possesses at any instant three points of contact with aa', the normals to these points always intersecting at the virtual centre. It would, in fact, be necessary to repeat the construction of Fig. 122, assuming two other portions of the outline of aa', and finding two new portions of the outline of bb', the centrodes, of course, remaining the same as before. It may be noted that while the relative motion of the centrodes is one of simple rolling, that of the two outlines is in general rolling and sliding combined.

63. Condition for Uniform Velocity Ratio.—We have seen in § 60 that when two bodies are in continuous contact and are capable of constrained relative motion, the normals at the points of contact must intersect at the virtual centre.

Consider now the case of three bodies (Fig. 123) of which

a and *b* turn around permanent centres O_{ac}, O_{bc}, with reference to the fixed body *c*, the bodies *a* and *b* being connected by higher pairing at a point of contact, *P*. The three bodies

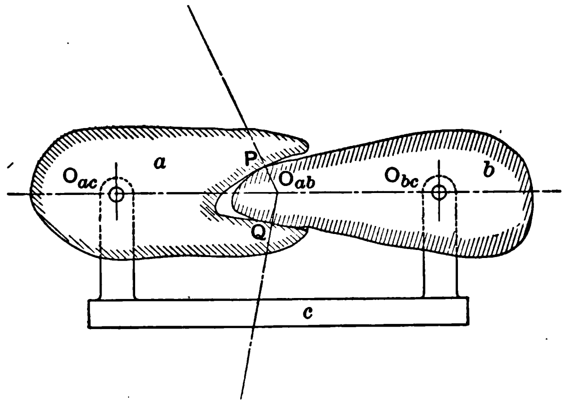

FIG. 123.

thus form a kinematic chain of three links, and we know that their virtual centres must lie in a straight line. Hence O_{ab} lies on the line joining O_{ac} and O_{bc}. But the normal at *P* must pass through O_{ab} if contact is to be continuous. The position of O_{ab} is thus fixed, and if *a* and *b* are to have one degree of freedom, the normal at any other point of contact, *Q*, must likewise pass through O_{ab}.

Next suppose that at the instant considered *a* has a certain angular velocity, ω_{ac}, with reference to *c*, in consequence of which any point on *a* will have an instantaneous linear velocity measured by the product $\omega_{ac}\times$radius. The point O_{ab} is a point common to *a* and *b*, and its linear velocity will be $\omega_{ac}\times\overline{O_{ac}O_{ab}}$, the direction being of course perpendicular to the radius, i.e., perpendicular to the line $\overline{O_{ac}O_{ab}}$. Knowing the linear velocity of O_{ab} we can find the angular velocity of *b*, which must of course be

$$\omega_{bc}=\frac{\text{linear velocity}}{\text{radius}}=\frac{\omega_{ac}\times\overline{O_{ac}O_{ab}}}{\overline{O_{ab}O_{bc}}}.$$

Hence

$$\frac{\text{angular velocity of } a}{\text{angular velocity of } b} = \frac{\omega_{ac}}{\omega_{bc}} = \frac{\overline{O_{ab}O_{bc}}}{\overline{O_{ab}O_{ac}}}.$$

In other words, the virtual centre O_{ab} divides the distance $\overline{O_{ac}O_{bc}}$ inversely in the proportion of the angular velocities of b and a with regard to c. From this important result it follows that if the angular velocity ratio for the bodies a and b is to be constant, O_{ab} must be a fixed point, in which case its path on each of the bodies a and b must be a circle. Uniform angular velocity ratio will then involve the rolling together of circular centrodes.

If we desire to find the angular velocity of a with regard to b (that of b with regard to c being known) we have only to suppose b fixed, and it follows that

$$\frac{\omega_{ab}}{\omega_{cb}} = \frac{\overline{O_{ac}O_{bc}}}{\overline{O_{ac}O_{ab}}}.$$

64. Wheel-gearing.—It may be noticed that if we form a and b in the shape of their centrodes we get a means of connecting two shafts (fixed to these bodies) with uniform angular velocity ratio, and the angular velocities, as we have just seen, will be in the ratio of the radii of the circular centrodes r_a and r_b, supposing that these centrodes roll together without slipping.

Thus in Fig. 124 we should have

$$\frac{\omega_{ac}}{\omega_{bc}} = \frac{QR}{PR} = \frac{r_b}{r_a}.$$

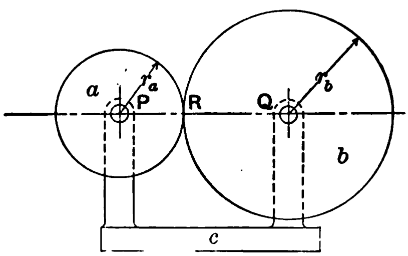

FIG. 124.

Such kinematic chains, used for the purpose of trans-

mitting motion from one shaft to another with any desired uniform or variable velocity ratio, are known as *wheel-trains* or *gear-trains;* the shafts, as we shall see in a later chapter, need not be parallel, but may intersect or even may not meet at all, the gearing being arranged to suit these conditions.

It is sometimes possible to use the actual forms of the centrodes (or, more strictly speaking, axodes) as the shapes of the gear-wheels. For example, the centrodes of the short links of an anti-parallel crank-chain might be replaced by elliptical gear-wheels, as we have already seen in § 57, and if these wheels rolled together without slipping, the shafts attached to them would at every instant have the same velocity ratio as the short links of the original chain. In the same way we can design many other forms of gear-wheels which when rolled together without slipping will transmit motion from one turning pair to another with some desired uniform or variable velocity ratio. Such wheels in the form of smooth axodes are not very useful in practice on account of their liability to slip; it is therefore usually necessary to provide the surface of each with teeth. These teeth have higher pairing, and their relative motion is in general combined rolling and sliding. The form of their profiles can therefore be determined by the method already explained in § 62, the two centrodes and the form of one profile being given. We shall return later to the question of the forms of wheel-teeth.

Wheels having such forms that their outlines are their own centrodes (or, more correctly, their surfaces are their own axodes), take a great variety of forms. It will be sufficient to notice here only cases in which the axes of rotation are parallel, and the planes of motion of the wheels therefore coincide. The elliptical wheels already mentioned afford one example, but other forms, usually termed *lobed wheels*, are occasionally met with. In every instance the point of

contact of the centrodes, i.e., the virtual centre of one wheel with regard to the other, must lie somewhere on the line of wheel centres. In a pair of lobed wheels, the difference between the greatest and least radii of the centrodes is called the *inequality;* in elliptical wheels the inequality is equal to the focal distance of either ellipse. Properly shaped lobed wheels working together have the same inequality. Thus in Fig. 125 the difference of PA and PB, the greatest

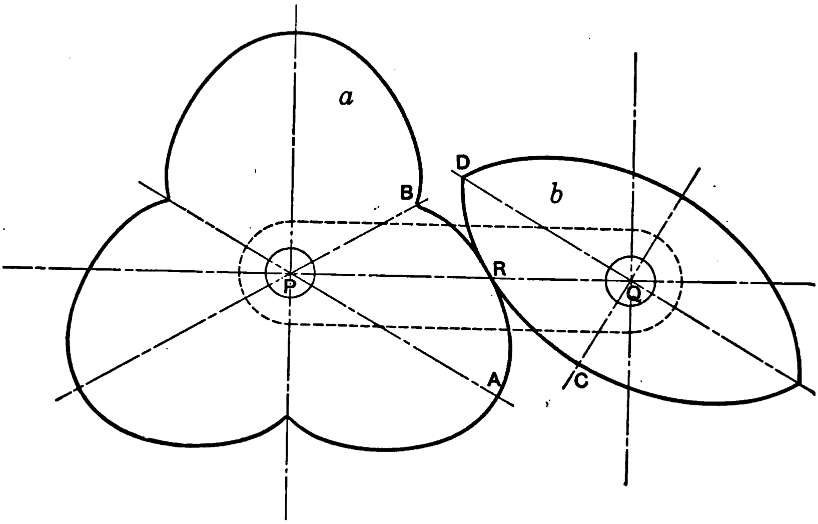

FIG. 125.

and least radii of the three-lobed wheel a, must be the same as $QD - QC$, the inequality of the two-lobed wheel b with which a gears. Further, the outlines of the wheels must be such that in every position of contact the point R lies somewhere on the line PQ, for if this were not so R could not be the virtual centre of a with regard to b.

In practice lobed wheels find only a very limited application, and the reader is referred to other works for information as to the shape to be adopted in any particular case.*

* See Rankine's Machinery and Millwork, p. 97; MacCord, Kinematics of Mechanical Movements, § 98.

65. Spur-wheels. — Wheel-gearing is most frequently employed to connect two shafts whose axes are parallel and whose angular velocities are in a constant ratio. It is sometimes sufficient to use *friction gearing* (in the form of smooth or grooved circular rollers), but gearing of the kind now to be discussed is in general provided with teeth lying parallel to the axes of the wheels, and is known as *spur-gearing*. The circular centrodes are called the *pitch circles* of the wheels, for it is around their circumferences that the teeth are set off.

Let a and b (Fig. 126) be portions of two spur-wheels gearing together with uniform velocity ratio. c is the fixed

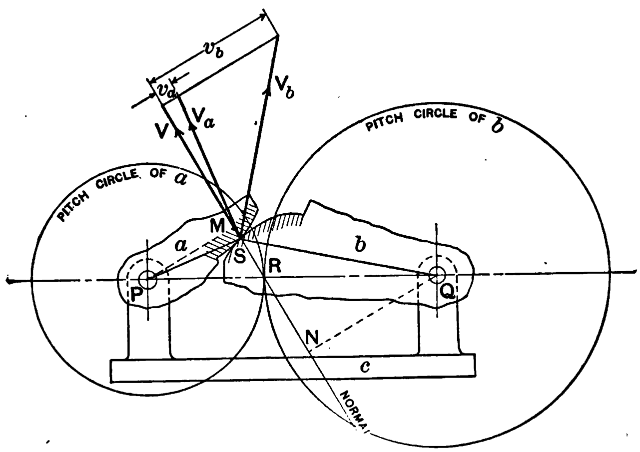

FIG. 126.

link of the chain, and P and Q are O_{ac} and O_{bc} respectively. The point $R(=O_{ab})$ is the point of contact of the centrodes or pitch-circles and is called the *pitch-point*.

Let S be the point at which the profile of the tooth formed on a is in contact with the profile of the tooth formed on b. Let SR be the common normal to the tooth profiles at S, the point of contact. At the instant considered, V, the velocity of S resolved along SR, must be the same whether we con-

sider S as a point on a or as a point on b. The actual motion of S (as a point on a) is that due to a velocity V_a in a direction at right angles to PS. Similarly the actual velocity of S (as a point in b) must be V_b, at right angles to QS. It is plain that if contact is to be maintained at S during the instant considered, V_a and V_b must have the same component V along the common normal to the two surfaces at S. We may in fact regard V_a as the resultant of two velocities, V along the normal and v_a perpendicular to it; similarly V_b is the resultant of V and v_b.

Evidently $v_b - v_a$ measures the speed at which the surface of b is sliding relatively to that of a; one object in a well-designed gear should be to make this sliding motion as small as possible, so as to minimize wear.

The angular velocity of a is measured by the ratio $\frac{V_a}{PS}$, which from the figure is seen to be equal to $\frac{V}{PM}$, where PM is the length of the perpendicular dropped from P on RS. In the same way $\frac{V}{QN}$ measures the angular velocity of b, and

$$\frac{\omega_{ac}}{\omega_{bc}} = \frac{V}{PM} \div \frac{V}{QN} = \frac{QN}{PM} = \frac{QR}{PR},$$

which is constant if R is a fixed point in the line PQ.

We thus see that for uniform velocity ratio the forms of teeth must be such that the common normal at the point of contact always passes through a fixed pitch-point, R, which divides the line of centres in the inverse ratio of the angular velocities, and is in fact O_{ab}. This important result has already been obtained in a more general manner in § 63.

Two kinds of curves at once suggest themselves as being suitable for wheel-teeth profiles, because their normals are easily found. These are involutes of circles and the vari-

ous cycloidal curves produced by rolling one circle, attached to which is a describing point, inside or outside the circumference of another base-circle. In the case of a rack, which may of course be looked upon as a wheel of infinitely large diameter, the base-circle is replaced by a straight line.

66. Involute Teeth.—Taking first the case of involute teeth, let AR and BR (Fig. 127) be the pitch-circles of a

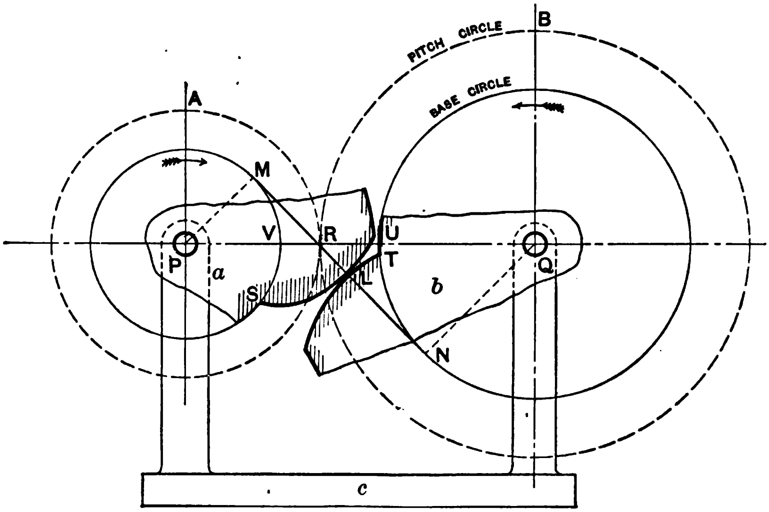

FIG. 127.

and b, a pair of wheels to be geared together with uniform velocity ratio. R is the pitch-point. Now let MS and TN be a pair of circles concentric with the pitch-circles, and let

$$\frac{PM}{QN}=\frac{PR}{QR}.$$

Draw MN, the common tangent to MS and TN; evidently MN passes through R. Next suppose that MN represents a flexible string wrapped round the circles MS and TN and kept stretched between them. The desired velocity ratio (represented by the fraction $\frac{QR}{PR}$) will correspond to the

rolling of BR on AR, and also to the relative movement f MS and TN, if connected by the string MN. Consider any point such as L, supposed to be fixed on the string. As the string unwraps from a, L will describe the curve SL on the wheel a, and SL will be an involute of the base-circle MS. Similarly, while the string is wrapping on to TN the point L will describe on the wheel b the involute LT of the base-circle TN. It is plain that the curves LT and LS must always touch at some point, L, on the line MN, which line is thus seen to be the path of the point of contact. Again, it is a property of the involute of a circle that the tangent, LN, drawn to the base-circle from any point L on the curve, is a normal to the curve at that point. The line MN is thus in every position of the wheels the common normal at the point of contact of the curves SL and LT, and it passes through the fixed pitch-point R. The condition for uniform velocity ratio is thus fulfilled, and if the teeth on a and b have their profiles formed on the curves SL, LT, they will work correctly together. The same curves would work together if the distance PQ were increased or diminished, for the common tangent MN would still divide PQ in the same ratio, and would still be the common normal at the point of contact.

In order to complete the outlines of the wheels the numbers of teeth must be decided. Evidently the length of tooth measured radially cannot be greater than UV, and must in practice be somewhat less. Further, in order that the wheels may work properly, a second pair of profiles must be commencing contact when the first pair cease to touch; in actual gearing at least two pairs of teeth are always in contact. This means that supposing the tooth profiles SL, LT are just ceasing to touch near N, a second pair must be touching at R, and a third pair preparing to begin contact at or near M; in other words, RM must be not less than the distance from the point where the pitch-circle cuts the face of one tooth to the point where it cuts the

face of the next. This distance, measured along the pitch-circle, is called the *pitch*, and the number of teeth on the wheel must be equal to the fraction (circumference of pitch-circle ÷ pitch). The pitch must plainly be the same for every wheel of a set gearing together, and we thus see that the numbers of teeth of wheels gearing together are proportional to the circumferences, or to the diameters, of their pitch-circles, and hence are inversely as their angular velocities.

The chief practical objection to the use of involute teeth is that the pressure between them acts obliquely along the line MN, instead of acting along a line perpendicular to UV. It is not possible to make involute wheels of only a few teeth without increasing this obliquity to an undesirable extent.

67. Cycloidal Teeth.—Cycloidal curves have the geometrical property that the normal to the curve at any point passes through the point of contact of the describing circle with the circle or straight line on which it is rolling. In Fig. 128 let AR and BR represent the pitch-circles of a pair of wheels which are to have cycloidal teeth, P and Q being the centres of the wheels. As the pitch-circles roll together during the motion of the wheels, imagine that a third circle, CR, pivoted at M, can roll in contact with the other two, and let L be a describing point on the circumference of CR, the three circles always touching at R. Suppose the three circles to move as shown by the arrows, and let S and T be the points on the pitch-circles of a and b respectively, which are in contact when L is at R. As the circle CR rolls on the outside of the circle BR, we may imagine that the describing point L traces on the wheel b the curve TL, which is therefore an epicycloid. In the same way L traces on a the hypocycloid SL, and the curves SL and TL will, of course, always be in contact at the point L. Since R is the point on the circle CR which is at rest relatively to the circles AR and BR (for the circles roll and do not slip), it follows that the direc-

tion in which L is moving at any instant relatively to the line of centres PQ, must be at right angles to the straight line LR, or, in other words, LR is the normal to the curves TL and SL at the point L. These cycloidal curves there-

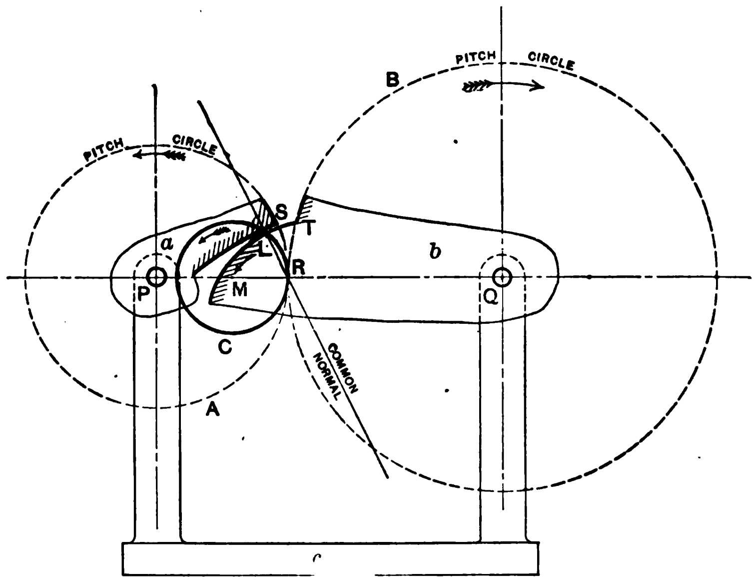

FIG. 128.

fore fulfil the condition for uniform velocity ratio of the two wheels, and this fact does not depend on the size of the describing circle CR.

The length and pitch of the teeth must be such that at least two pairs are always in contact, and the teeth are spaced out along the pitch-line exactly in the way described for involute teeth. Notice that the path of the point of contact for cycloidal teeth is an arc of the describing circle (arc LR in Fig. 128).

Cycloidal teeth in practice are almost invariably drawn by the use of two describing circles. Fig. 129 shows a pair of cycloidal teeth profiles just commencing contact at L_1, and just ceasing to touch at L_2. The curves L_1T_1 and L_1S_1

are described by the circle C_1R, while L_2T_2 and L_2S_2 are drawn by a describing point on C_2R. Note that the whole path of the point of contact is L_1RL_2. In the case of a single pair of wheels the circles C_1R and C_2R need not be of the same diameter, but if a set of wheels is to be made, any wheel of which is to gear with any other, the wheels must not only have teeth of the same pitch, but these teeth

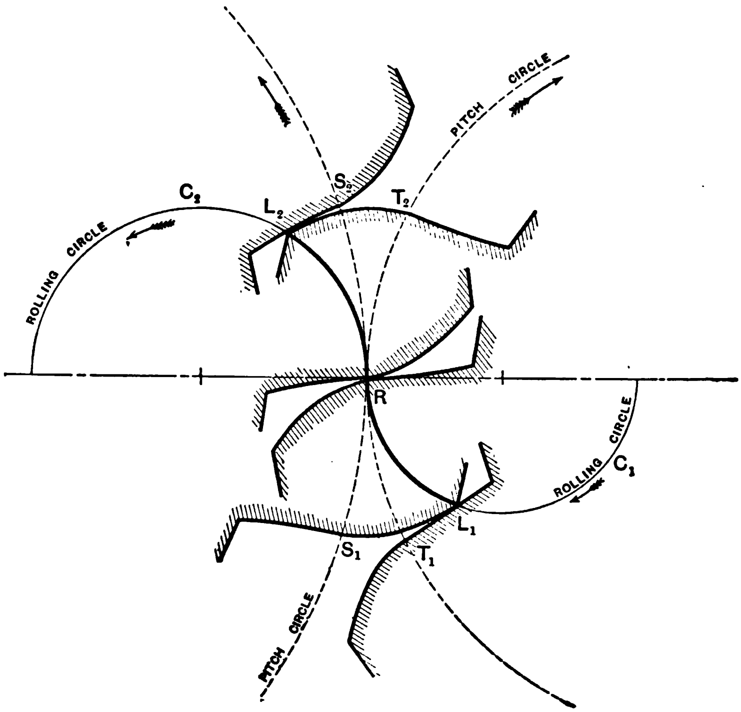

FIG. 129.

(if cycloidal) must all be drawn with describing circles of the same diameter.

Involute and cycloidal curves are, of course, not the only ones that can be chosen for the purpose of giving uniform velocity ratio. They are, however, easily drawn, and for this reason are generally used.

We might have chosen arbitrarily a form for the teeth

of one wheel, and then, having given the centrodes (pitch-circles), we might have determined the proper form for the teeth of the second wheel by the method of § 62. In some cases this would give a form impossible for constructive reasons, although correct so far as relative velocity is concerned. For instruction as to the design and proportion of wheel-teeth the reader should consult Unwin's "Machine Design," Vol. I, chapter X, or other works on the same subject.

CHAPTER VIII.

WHEEL-TRAINS AND MECHANISMS CONTAINING THEM. CAMS.

68. Simple and Compound Wheel-trains.—The determination of the velocity ratio in such a wheel-train as that of Fig. 130 involves no difficulty, for it is plain that one or

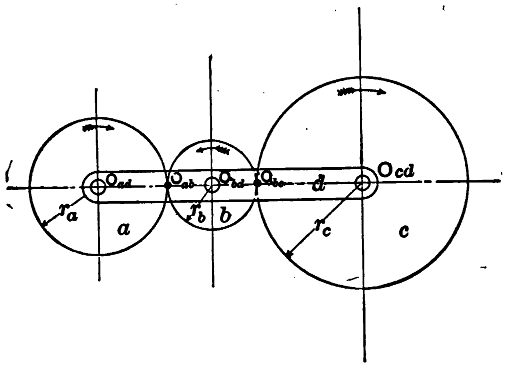

FIG. 130.

more intermediate wheels (as b) will not affect the numerical value of the velocity ratio of the first and last wheels. The linear velocity of the pitch-line of every wheel is the same, and the angular velocity ratio of the first and last, therefore, only depends on their own diameters, so that $\frac{\omega_{ad}}{\omega_{cd}} = \pm\frac{r_c}{r_a}$, the sign depending on the number of idle wheels. Intermediate or *idle* wheels thus simply reverse the direction of motion. When all the wheels in the train have external contact, the angular velocity ratio of the first wheel to the last has a positive value (or, both wheels turn in the same

sense) if the number of axes is odd, while an even number of axes gives the velocity ratio a negative value. More complex wheel-trains, however, require further consideration. In Fig. 131 we have a compound spur-wheel mechanism of

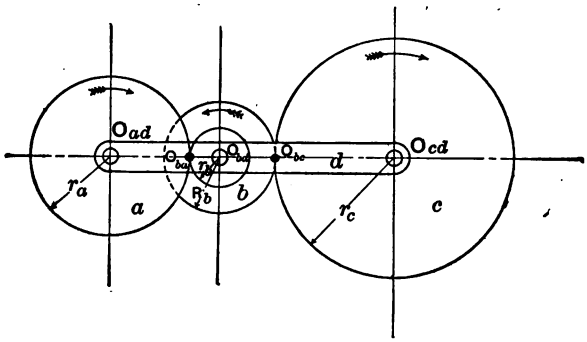

FIG. 131.

four links, d being fixed, while b consists of two wheels rigidly connected and turning on the same axis.

Let r_a, r_b, R_b, r_c, be the radii of the pitch-circles, then from § 64 we have

$$\frac{\omega_{ad}}{\omega_{bd}}=\frac{r_b}{r_a}.$$

Also,

$$\frac{\omega_{bd}}{\omega_{cd}}=\frac{r_c}{R_b}.$$

Hence

$$\frac{\omega_{ad}}{\omega_{cd}}=\frac{r_b \times r_c}{r_a \times R_b}=N.$$

Suppose a to be the driving-wheel, while c is the driven one; we see that the above result may be expressed by saying that

$$\text{velocity ratio}=\frac{\text{revolutions of driving-wheel}}{\text{revolutions of driven wheel}}$$

$$=\frac{\text{product of radii of followers}}{\text{product of radii of drivers}}.$$

Instead of radii we might evidently put numbers of teeth.

It would be easy to find a single pair of wheels having the same velocity ratio as the given train. For example, if we had a pair of wheels, A and C, such that

$$r_A - r_C = r_a + r_b + R_b + r_c \quad \text{and} \quad \frac{r_A}{r_C} = \frac{r_a \times R_b}{r_b \times r_c},$$

these would have the same velocity ratio and the same distance from centre to centre. The point of contact of their pitch-circles would divide the distance $O_{ad}\ O_{cd}$ externally in the proportion of the angular velocities of a and c, and would in fact be the point O_{ac}.* Hence in Fig. 131 we have only to divide the line of centres, graphically or otherwise, in the proper ratio to find the sixth virtual centre.

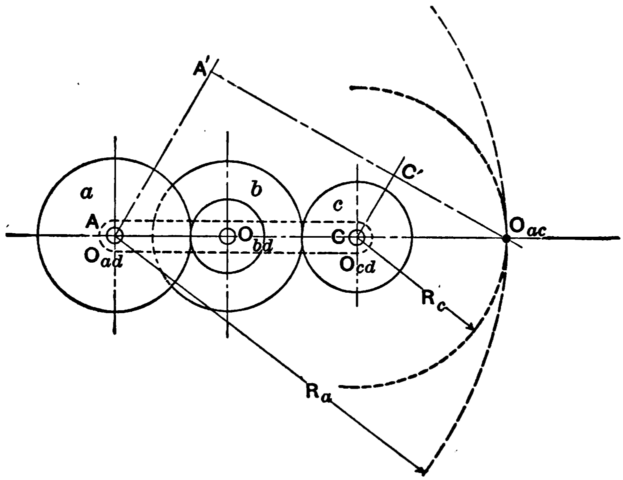

FIG. 132.

In doing this (as in working all problems connected with wheel trains), note must be taken of the sign of the velocity ratio, which depends on the presence or absence of *annular wheels* (i.e., wheels having internal contact), or of idle wheels, and also on the number of axes in the train. Take, for example, the two trains shown in Figs. 132 and 133, in

* For graphic methods of determining virtual centres of wheel-trains, see Kennedy, Mech. of Machinery, note to Chapter VI.

the first of which suppose $r_a=2$, $r_b=1$, $R_b=2$, $r_c=1.5$, so that the velocity ratio in Fig. 132 has the value

$$N=\frac{\omega_{ad}}{\omega_{cd}}=+\frac{1.5\times 1}{2\times 2}=+\frac{3}{8}.$$

In Fig. 133 we have a simple train having exactly the same numerical value for its velocity ratio (since $\left.\frac{r_c}{r_a}=-\frac{3}{8}\right)$, but in this case the negative value must be adopted, since the wheels a and c turn in opposite senses. In Fig. 132 O_{ac} may be found by drawing AA' CC' parallel to one another, and of lengths 8 and 3 respectively, to any convenient scale. The intersection of $A'C'$ with the line of centres fixes O_{ac}. Evidently the given train might be replaced by a pair of wheels of radii R_a and R_c, the larger being annular, having their centres at A and C, and their pitch-circles touching at O_{ac}, as shown by the dotted arcs. Again, in Fig. 133

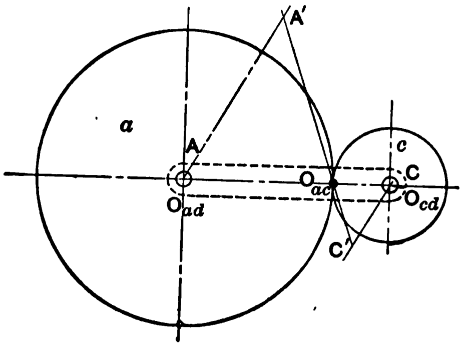

FIG. 133.

AA' CC' must be drawn parallel, but in opposite senses, so as to allow for the negative velocity ratio, and O_{ac} is, of course, the point of intersection of AC and $A'C'$.

It is by no means necessary to have the centres of all the wheels of a train in one straight line. The back-gear of a lathe, for example, is an instance of a compound *reverted train* in which the centres of the first and last wheels coincide.

This arrangement makes no difference in the numerical value of the velocity ratio, and is simply adopted for convenience in construction.

69. Epicyclic Gearing.—In the above examples of wheel-trains we have supposed the frame carrying the wheels to be the fixed link. Wheel gearing is often employed in which one of the wheels is the fixed link and the frame or arm carrying the remaining wheels is movable. Such gearing is called *epicyclic*, and we proceed to discuss some of its simpler cases.

We take first the mechanism of Fig. 133, but suppose a to be fixed, while d is rotated in a clockwise or positive sense (Fig. 134). Let N be the velocity ratio of the train, i.e., let

$$N = -\frac{r_a}{r_c} = \frac{\omega_{cd}}{\omega_{ad}}.$$

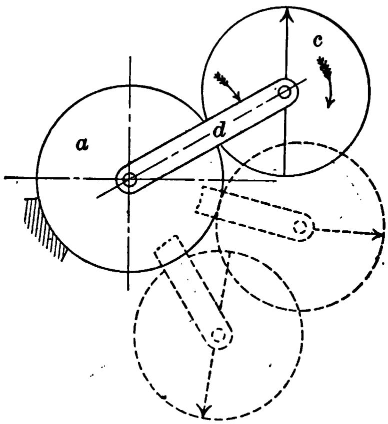

FIG. 134.

Plainly, if we consider ω_{da} as being positive in sign, then ω_{ad} must be negative, hence

$$\omega_{ad} = -\omega_{da}.$$

Now in any case where two bodies, c and d, have motion relatively to a third, a, which is fixed, any angular movement of c relatively to a may be looked on as the algebraic

sum of the motions of c relatively to d and of d relatively to a. Thus

$$\begin{aligned}\omega_{ca} &= \omega_{cd} + \omega_{da} \\ &= +\omega_{cd} - \omega_{ad} \\ &= -\omega_{ad}\left(1 + \frac{r_a}{r_c}\right),\end{aligned}$$

or

$$\frac{\omega_{ca}}{\omega_{da}} = 1 - N,$$

where N is itself a negative quantity. A numerical example may, perhaps, make this clearer. Suppose the wheels a and c to have 100 teeth and 90 teeth respectively; these teeth have the same pitch, and we can, of course, take the ratio of the numbers of teeth instead of the ratio of the diameters or radii of the pitch-circles. Thus $\frac{r_a}{r_c} = \frac{10}{9}$; in other words, supposing d to be fixed, while a makes one revolution with

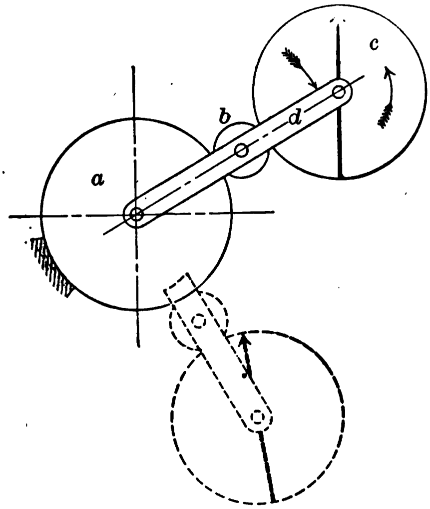

FIG. 135.

regard to d, c would make $1\frac{1}{9}$ in the opposite sense. Now suppose that in a certain time a makes -1 revolution, c making $+1\frac{1}{9}$, while d is at rest. Cause the whole mechanism to execute $+1$ rotation in the same time around O_{ad};

this brings a to rest, makes d perform $+1$ revolution, and therefore gives c $1\frac{1}{9}+1=2\frac{1}{9}$ revolutions in the same sense as that of the arm.

If an idle wheel, b, had been interposed between a and c, as in Fig. 135, we should have had $N=+\frac{r_a}{r_c}=\frac{\omega_{cd}}{\omega_{ad}}=$ a positive quantity, and $\omega_{ca}=\omega_{da}+\omega_{cd}$, whence $\frac{\omega_{ca}}{\omega_{da}}=1+\frac{\omega_{cd}}{\omega_{da}}=1-N$, as before. With the numbers of teeth, as in the example just given, and the train arranged as in Fig. 135, we should have, if $N=+1\frac{1}{9}$,

$$\frac{\omega_{ca}}{\omega_{da}}=1-1\tfrac{1}{9}=-\tfrac{1}{9};$$

i.e., for each revolution of the arm, c makes $\frac{1}{9}$ revolution in the reverse sense.

Fig. 136 represents a compound epicyclic reverted train. Let n_a, n_{b_1}, n_{b_2}, n_c be the numbers of teeth in the wheels

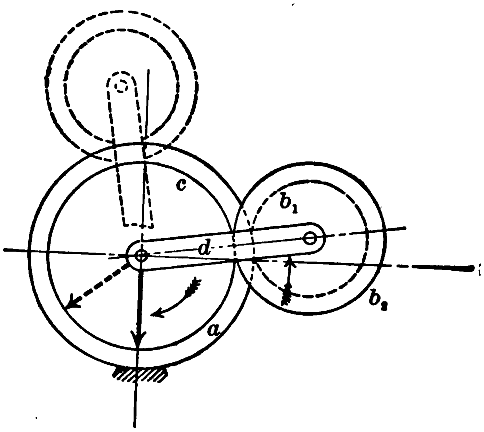

FIG. 136.

a, b_1, b_2, and c respectively. Evidently, if the pitch of both pairs is the same, $n_a+n_{b_1}=n_{b_2}+n_c$. The velocity ratio of the train will be positive and has the value

$$N=\frac{\omega_{cd}}{\omega_{ad}}=\frac{n_a\times n_{b_2}}{n_{b_1}\times n_c},$$

hence $\omega_{cd} = -(N \times \omega_{da})$.

Further, when a is the fixed link

$$\omega_{ca} = \omega_{cd} + \omega_{da} = \omega_{da}(1 - N).$$

Thus, for instance, suppose a has 30 teeth, b_1 has 15, and b_2 and c have respectively 20 and 25; then

$$N = +\frac{30 \times 20}{15 \times 25} = +1.6 = \frac{\omega_{cd}}{\omega_{ad}},$$

and $$\frac{\omega_{ca}}{\omega_{da}} = 1 - N = -0.6.$$

Thus c will make 0.6 revolution for each revolution of the arm, but in the opposite sense. Such a train might evi-

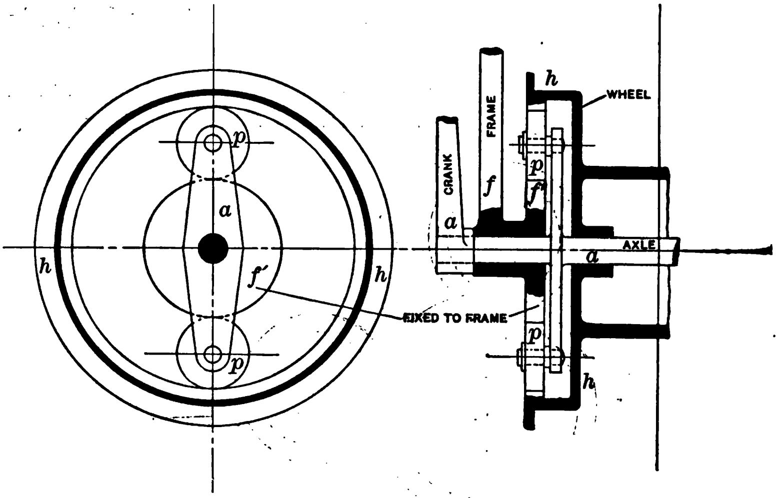

FIG. 137.

dently be arranged to give c a very slow rotary motion, say $\frac{1}{10,000}$ revolution for each revolution of the arm d.

As an example of an epicyclic reverted gear containing an annular wheel the wheel-train used in certain front-driving bicycles* may be given. In Fig. 137 ff represents

* See also the Weston triplex pulley-block described in § 78, Chapter IX.

part of the front fork of the bicycle, to which is rigidly attached the central pinion f'. The arms a, secured to the axle and cranks, carry one or more planet-wheels, p, gearing with the central pinion and with an annular wheel formed on the inside of the hub, h, of the driving-wheel. Suppose this wheel h has 60 teeth, while p has 15 and f' has 30; it is plain that $n_h = n_f + 2n_p$ if the wheels are to gear together and the wheel h is to be coaxial with f'.

Now if a were the fixed link,

$$\frac{\omega_{ha}}{\omega_{fa}} = -\frac{30}{60} = -\tfrac{1}{2};$$

therefore $$\omega_{ha} = -\tfrac{1}{2} \times \omega_{fa} = \tfrac{1}{2}\omega_{af}.$$

The ratio to be determined is the number of revolutions of the wheel h per revolution of the crank a; this is the same quantity as

$$\frac{\omega_{hf}}{\omega_{af}} = \text{velocity ratio of } h \text{ and } a.$$

Now $$\begin{aligned}\omega_{hf} &= \omega_{ha} + \omega_{af} \\ &= (\tfrac{1}{2} + 1)\omega_{af}.\end{aligned}$$

Hence $$\frac{\omega_{hf}}{\omega_{af}} = +1.5;$$

in other words, the wheel will make $1\frac{1}{2}$ revolutions for each revolution of the crank, and in the same sense. A bicycle having a driving-wheel 44 inches diameter would therefore be geared to 66 inches with this arrangement.

70. Mechanisms Containing Wheel-trains.—Mechanisms are of common occurrence in which wheel trains form part of chains containing also sliding and turning pairs. Fig. 138 shows diagrammatically a "sun-and-planet" gear containing an annular wheel, forming part of a mechanism containing a slider-crank chain.

The crank a is able to rotate about the point O_{ad} with reference to a fixed frame d, and pairs with a link b, forming the connecting-rod in a slider-crank chain, of which c is the

sliding block. On b, however, is formed a spur-wheel whose pitch-circle has its centre at O_{ab}. The spur-wheel gears with an annular wheel e whose pitch-circle has its centre at O_{ad}. The virtual centres are marked on the dia-

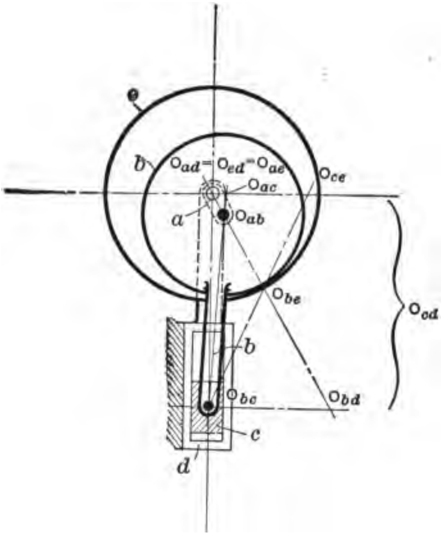

FIG. 138.

gram. We wish to find the number of revolutions of the annular wheel e for each revolution of the crank a.

As in previous examples, let N be the velocity ratio of the wheel-train; i.e., let

$$N = \frac{\omega_{ea}}{\omega_{ba}} = \frac{n_b}{n_e} = \frac{r_b}{r_e}.$$

Plainly N will be a positive fraction in this case. We note that during the action of the mechanism the average value of ω'_{bd} is zero, for b simply swings to and fro, and a line marked on it describes equal angles right and left from its mid-position. Hence we may say that on the average

$$\omega_{ab} = \omega_{ad} + \omega_{db} = \omega_{ad};$$

that is, we may consider the angular velocity of a and b instead of that of a and d. The two would always be exactly

equal if b always remained parallel to itself, i.e., if the connecting-rod were infinitely long.

Now
$$\omega_{ed} = \omega_{ad} + \omega_{ea}.$$
$$\frac{\omega_{ed}}{\omega_{ad}} = 1 + \frac{\omega_{ea}}{\omega_{ad}}$$
$$= 1 - \frac{\omega_{ea}}{\omega_{da}}$$
$$= 1 - \frac{\omega_{ea}}{\omega_{ba}}$$
$$= 1 - N.$$

For example, suppose e had 100 teeth while b had 95, so that $N = +0.95$; then for each revolution of the crank a, e would make $1-0.95=0.05$ revolution in the same sense. This mechanism is actually used as gearing for a capstan driven by a hydraulic engine, b being attached to the connecting-rod, while the capstan barrel is attached to e.

As another example of a mechanism containing a wheel-train we may take the wheel crank-chain of Fig. 139, which

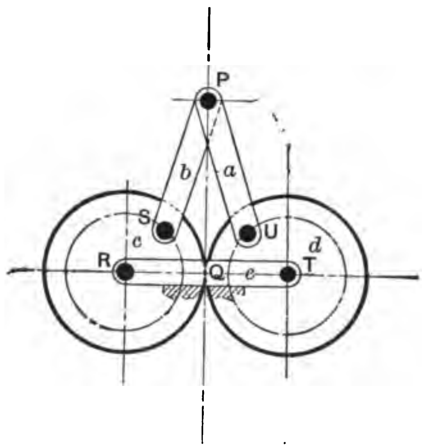

FIG. 139.

is formed by combining a simple wheel chain with an open crank-chain of five links.

If the lengths of the links a and b, and also those of c

and d, are equal, as in the figure, we obtain Cartwright's straight-line motion, in which the point P describes a straight-line path passing through Q. The purpose of the two spur-wheels is to close the five-link chain, whose motion would otherwise be unconstrained.

In Figs. 140a and 140b we have a slider-crank chain in which spur-wheels can be used for a somewhat similar purpose. Consider a slider-crank chain in which the connect-

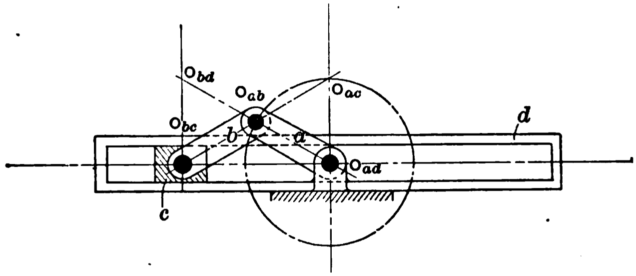

FIG. 140a.

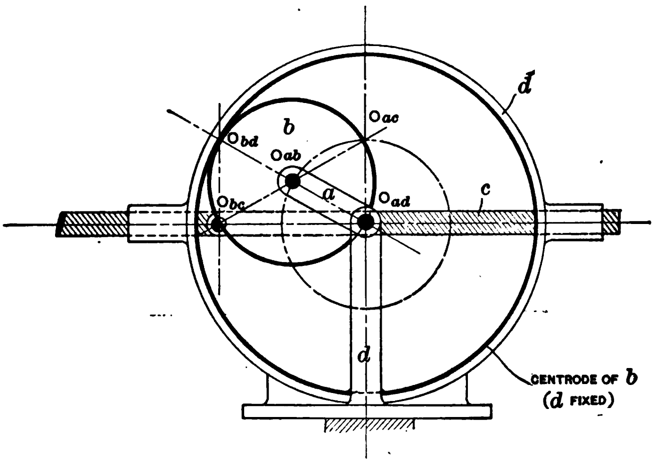

FIG. 140b.

ing-rod b is made equal in length to the crank a. (Compare Figs. 60 and 91.) With these proportions it is possible for the stroke of c to be (1) either twice the length of the crank,

as in the ordinary slider-crank chain, or (2) four times the length of the crank, in which case O_{bc} must travel on past O_{ad}. A third possibility is that O_{bc} and O_{ad} may remain coincident, in which case *b* and *a* move together, and the mechanism will reduce to a pair of elements.

Notice in Fig. 140*a* that since the length of *b* is equal to the length of *a* and the angle $O_{bd}O_{bc}O_{ad}$ is a right angle, the points $O_{bd}O_{ad}O_{cb}$ must lie on the circumference of a circle whose radius is the length of *a* or the length of *b*. Since $O_{bd}O_{ad}$ is a diameter of this circle, it follows that O_{bd} remains always at the same distance from O_{ad}, and the centrode of *b* with regard to *d* is a larger circle whose radius is twice the length of *a*. If now (Fig. 140*b*) we attach to *d* an annular wheel whose pitch-circle is the centrode of *b* with regard to *d*, and if we fix to *b* a spur-wheel whose pitch-circle is the centrode of *d* with regard to *b*, these wheels will gear together, and will compel O_{bd} to remain always at a fixed distance from O_{ad}. If these wheels were not provided we should have a change-point at the instant when O_{bc} passes O_{ad}, but if the virtual centre O_{bd} is compelled to remain at a fixed distance from O_{ad} by the action of the spur-wheels, O_{bc} is compelled to continue its travel, and the mechanism is not permitted to change. Obviously this arrangement is really a case of pair-closure at a change-point. (Compare the examples in § 59.) The only really essential portions of the wheels are therefore those teeth which are in gear while O_{bc} is passing O_{ad}.

71. Cam-trains.—The name cam-train is applied to mechanisms containing a rotating disc (generally non-circular) or a sliding plate, the profile of which forms one element of a higher pair and gives some desired periodic motion to the second element of the pair. Such a cam-pair may be closed by forming one element into the geometrical envelope of all possible positions of the other element. Mechanically cam-pairs usually possess the disadvantage of small wearing surface and rapid wear, common in higher pairs. Almost invariably force-closure is necessary to make up for the

looseness of fit following on wear. A cam-train is in general a mechanism of three links; such, for example, is the cam-train found in the stamp-mill used for crushing hard ores (Fig. 141).

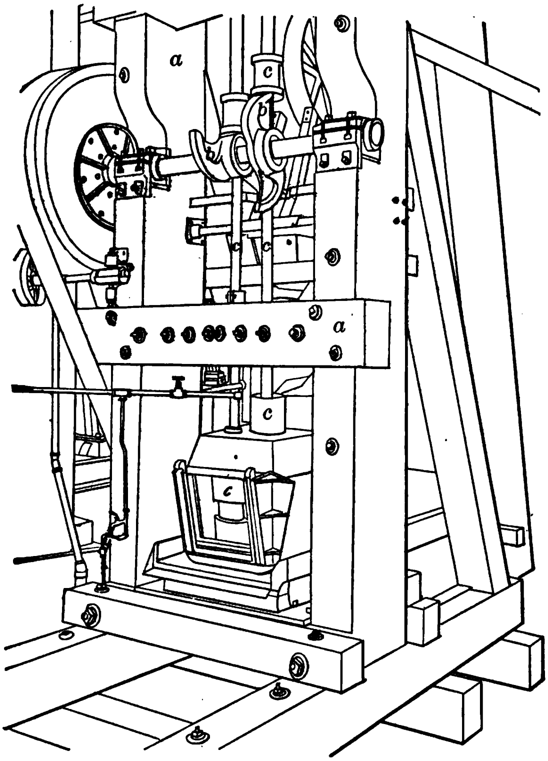

FIG. 141.

A rotating shaft carries the cams *bb*. These successively lift and let fall the stamps *cc*, which are guided by means of the framework *a*. It will be noted that the cam-pair *bc* is force-closed by the weight of the stamp itself, and also that the form of the cam is such that in any position during the upward stroke it is touched by the horizontal under surface of the collar on the stamp-rod. This fact has to be consid-

ered in selecting the form of the cam, for it is obvious that during all the upward movement the point of the cam surface which touches the collar must be at a higher level than any other point on the cam surface. It will be found that with such a cam it is not possible to give the collar any arbitrary position for any given angular position of the cam.*

In many cases the cam has to determine the position of a point, instead of a flat surface, on the *follower* or link driven by the cam. This point is then usually taken as the centre of a roller or pin with which the cam engages; and within certain limits imposed by constructive considerations, any desired continuous change of position can be given to the follower by suitably choosing the form of the cam profile.

72. **Rotating Cams.**—The action of cam-trains will be most easily understood by the study of a few examples. To produce a given form of reciprocating motion along a straight or curved line we may employ either a rotating, a sliding, or a cylindrical cam. The first example (Fig. 142) will be that of a rotating cam designed to give its follower a reciprocating motion along a straight line passing through the cam centre, the velocity being uniform throughout both strokes if the cam rotates with uniform angular velocity. The mechanism is somewhat similar to that of Fig. 141, and consists of a cam c (whose form is to be determined), a guiding frame a, and a follower b, which is to slide with the periodic motion specified above. The end of the follower is provided with a roller, for the sake of lessening friction.

Since the cam rotates uniformly, while the follower moves with uniform velocity, the cam describes equal angles while the follower traverses equal distances. Plainly the outline of the curve required will be such that successive radii making equal angles with one another have a constant difference in length. All we have to do, in fact, is to divide the path of A into any convenient number of equal parts,

* See Kennedy, Mech. of Machinery, p. 154.

say six, and to divide the half-revolution of the cam into the same number of equal angles. If CA_0 is the least distance of the centre of the roller from the centre of the cam, and CA_1, CA_2, etc., are the distances after one, two, etc., twelfths of a revolution, we then make $Ca_1 = CA_1$, $Ca_2 = CA_2$, and so on. The curve drawn through a_1a_2 will be recog-

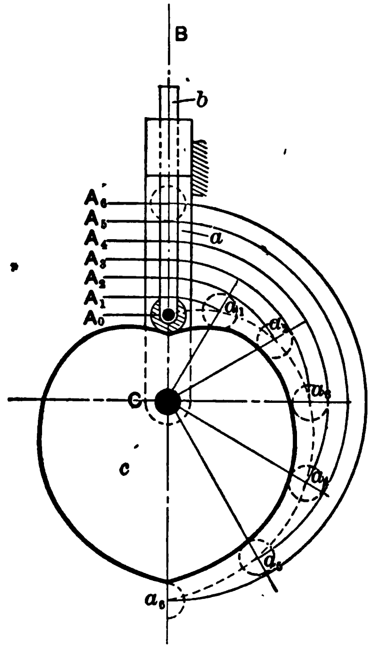

FIG. 142.

nized as an Archimedean spiral, and the same kind of curve will, of course, be found for the remaining half of the cam. In this curve let $CA_0 = r_0$, and let r be any radius-vector of the curve, then $r = r_0 + m\theta$, where θ is the angle the radius-vector makes with CB and m is a constant. The real outline of the cam itself is not the dotted line $a_1a_2a_3$, but the full line drawn so as to touch a series of circles whose centres lie on a_1a_2, etc., and whose diameters are all equal to that of the roller on the follower.

A cam frequently has to actuate a point on a lever, which of course moves in the arc of a circle. An exactly

similar construction in this case gives the form of the cam, but the points A_0, A_1, . . . , will now be placed along a circular path instead of along a straight line.

As a more difficult case, let us consider the form to be given to a cam arranged to move a follower with uniform *acceleration* during one half of a revolution, after which the

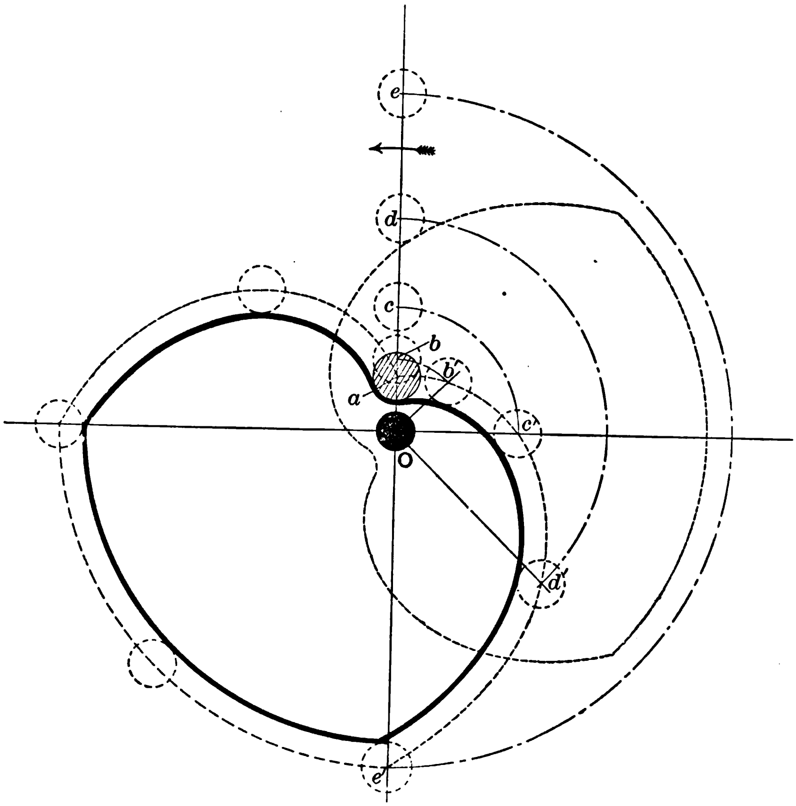

FIG. 143.

sliding piece remains at rest during a quarter of a revolution and returns with uniform velocity during the remaining quarter revolution, the cam rotating uniformly.

In Fig. 143 let the points *a*, *b*, *c*, *d*, *e* correspond to the positions of the centre of the pin on the reciprocating piece

at equal intervals of time during one half revolution, while the distance ae is the length of stroke of the reciprocating piece. Since the upward stroke is to be made with uniform acceleration, the distance $ac = 4ab$, while $ad = 9ab$, and $ae = 16ab$ (where ab is the distance moved in one eighth of a revolution). Let O be the centre of the cam. Then, starting with the sliding-piece in its lowest position, when the cam has turned through a quarter of a revolution, the line

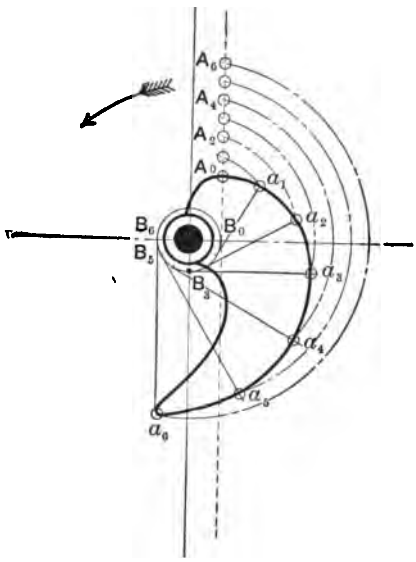

FIG. 144.

Oc' will coincide with Oc. Thus at the instant when the sliding-pin has its centre at c, the radius Oc' will be vertical, and we must set off a distance $Oc' = Oc$. By a similar construction other points on the curve, such as b', d', etc., are obtained. The profile of the cam itself is obtained by drawing a curve (shown in full lines) at a uniform distance from $ab'c'd'e'$ equal to the radius of the pin on the sliding-piece.

We may now consider the construction to be adopted if the line of motion of the follower-point does not pass through

the centre of rotation of the cam. In Fig. 144, let $A_0A_1A_2\ldots$ be successive positions of the follower-point, corresponding to successive equal angles described by the cam-shaft, and let the line $A_0A_1A_2\ldots$ produced be a tangent to a small circle $B_0B_1B_2B_3\ldots$ described about the cam centre.

As the cam rotates it is seen that such a line as B_6a_6 drawn touching the small circle will take up the position $B_0A_0A_6$ when it becomes vertical. Hence the point a_6 will be found by drawing a circle with O as centre, and radius OA_6 so as to cut B_6a_6; then B_6a_6 is the line which coincides with B_0A_0 at the time when the follower-point is at A_6.

If the distance A_0A_1 is equal to the arc B_0B_1, and the distances A_1A_2, B_1B_2, are equal, and so on, it is evident that the curve a_1a_2 is an involute of the base-circle $B_0B_1B_2B_3\ldots$ Such curves are generally used for the cams of an ore-crushing stamp-mill. In an involute the tangent to the base-circle is a normal to the curve (see § 61); hence in any position of an involute cam the point lying on the vertical line B_0A_6 will touch a horizontal line corresponding to the under surface of a collar on the stamp-rod.

73. Sliding and Cylindrical Cams.—The form of a *slid-*

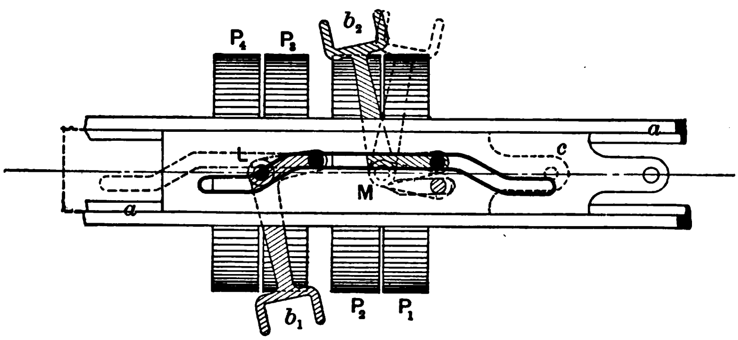

FIG. 145.

ing cam to obtain any desired kind of periodic motion is easily determined. Fig. 145 shows the arrangement of a cam of this kind used for giving the requisite motion to the

belt-shifting gear of a planing-machine. The cam c here takes the form of a slotted plate, sliding in a frame or guide a. Two bell-crank levers, b_1 and b_2, pivoted to a at L and M, carry follower-pins which work on the slot in c. The longer arms of the levers are provided with forks for shifting the belts as required from the fast to the loose pulleys of the planing-machine. The length and form of the slot in the cam are such that when c is in its extreme position to the left, as shown by dotted lines, b_2 is thrown to the right, and its belt runs on the fast pulley P_1, while the fork b_1 is also inclined to the left and its belt runs on a loose pulley P_3. On moving the cam from left to right it will be seen that b_2 first moves its belt on to the loose pulley P_2, and afterwards b_1 moves the second belt on to the fast pulley P_4. In this way it is impossible for both belts to be on fast pulleys at the same time. The belts drive the pulleys P_1 and P_4 in opposite senses, hence the action of the gear is to reverse the motion of the shaft to which the pulleys are attached. The motion of the planing-machine table is thus also reversed.

Fig. 146 shows a *cylindrical cam*, in which the cam profile

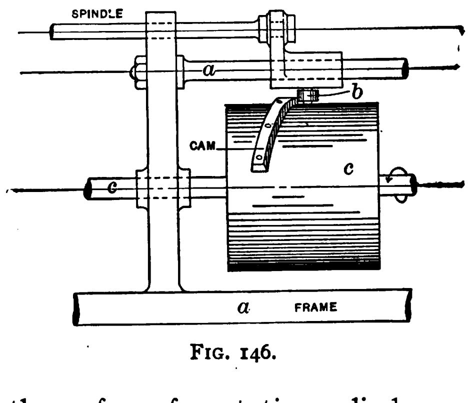

FIG. 146.

is traced on the surface of a rotating cylinder and the line of motion of the follower is parallel to the axis of the cylinder. The figure shows how such cams are employed in certain

automatic screw-making machines for the purpose of giving endwise motion to the rotating spindles carrying the work. The working profile of the cam is the edge of a strip secured to the surface of the cylinder *c* by screws; a series of these strips may evidently be arranged so as to give any desired periodic range of rest and motion to the carriage of the rotating spindle. In this case the roller on the follower *b* is kept pressed against the cam edge by a spring or other

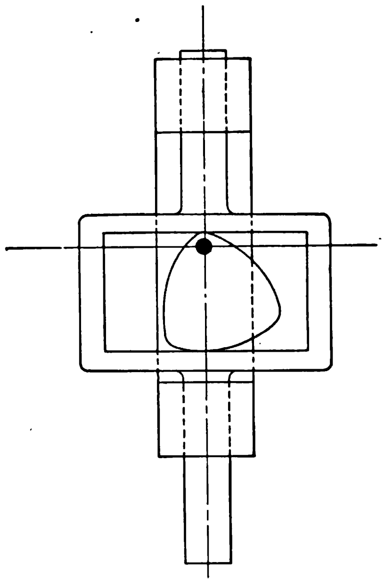

FIG. 147.

suitable means. The mechanism is thus *force-closed.* It would, however, be quite easy, by attaching two strips, to form a groove in which the follower-roller would work; the mechanism would then have *pair-closure.*

A rotating cam of the kind shown in Fig. 143 could be closed in the same manner if a groove were formed on the flat surface of the cam-plate, engaging with a pin or roller attached to the follower. Or, as an alternative, the cam may be formed with a figure of constant breadth, in which

case it can have constrained motion with relation to a follower provided with a pair of parallel faces, as in Fig. 147.

Cams which are pair-closed are often called positive-motion cams.

74. Velocity Ratio in Cam-trains.—Let Fig. 148 represent a cam-train of three links; we wish to find the velocity ratio of the train, i.e., the ratio

$$\frac{\text{linear velocity of follower}}{\text{angular velocity of cam}}.$$

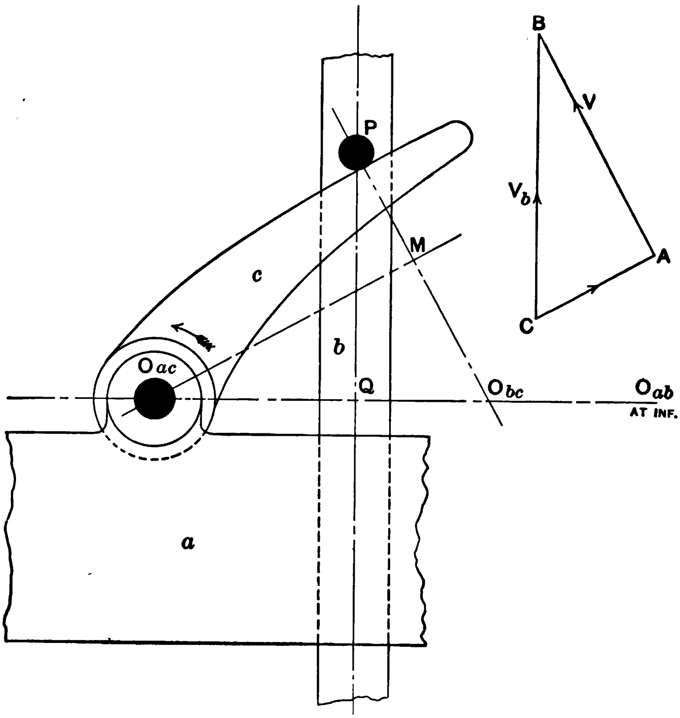

FIG. 148.

Find O_{ac}, the centre about which the cam rotates, and also O_{ab}, which in this case is at an infinite distance, since the follower moves in a straight line, QP. If the follower were a lever turning about a fixed centre, the line $O_{ac}O_{ab}$

would be as easily determined. The third virtual centre O_{bc} must be the point where the common normal to the cam and follower at their point of contact cuts this line. Let

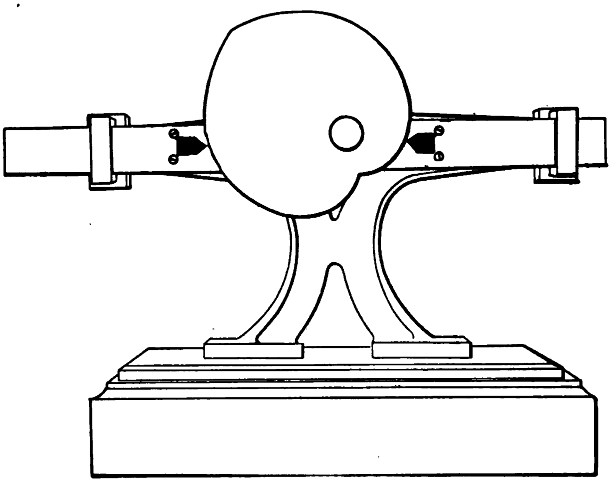

FIG. 149*a*.

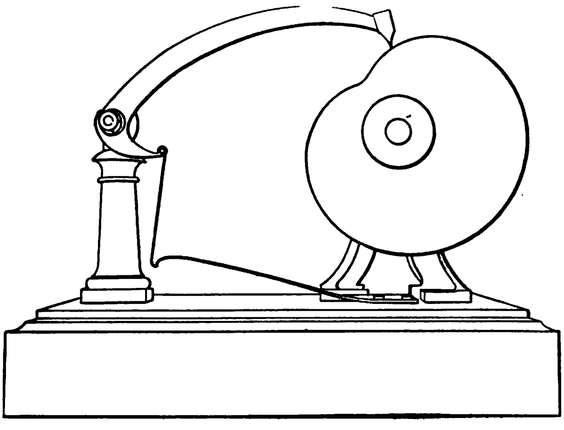

FIG. 149*b*.

V be the common velocity of the bodies b and c at their point of contact; its direction will be parallel to the line MP. Draw a triangle of velocities ABC, in which AB represents V, CB represents V_b, and AC represents the velocity of

sliding of c on b in a direction parallel to the common tangent at the point of contact. Draw $O_{ac}M$ parallel to this tangent, and therefore perpendicular to PM.

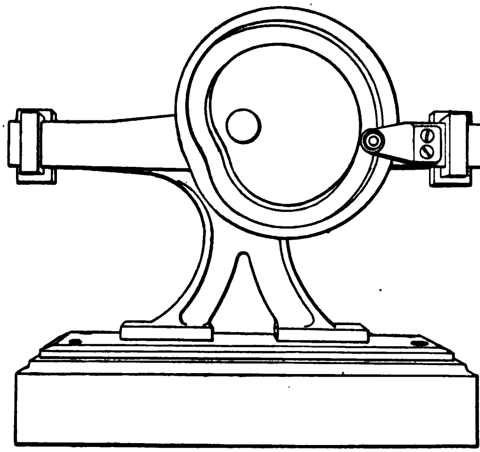

FIG. 149*c*.

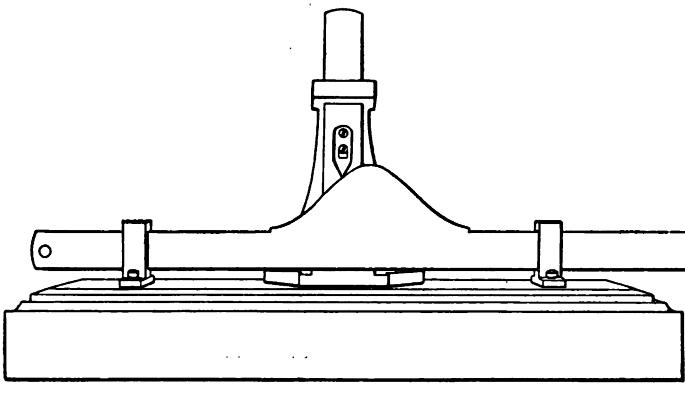

FIG. 149*d*.

Then the angular velocity of the cam

$$\omega_{ca} = \frac{V}{O_{ac}M}$$

and the linear velocity of the follower along QP

$$V_b = V \cdot \frac{BC}{BA} = V\frac{\overline{O_{ac}O_{bc}}}{O_{ac}M}.$$

Hence the velocity ratio of the pair is

$$\frac{V_b}{\omega_{ca}} = \frac{V \cdot \overline{O_{ac}O_{bc}}}{\overline{O_{ac}M}} \cdot \frac{\overline{O_{ac}M}}{V} = \overline{O_{ac}O_{bc}}.$$

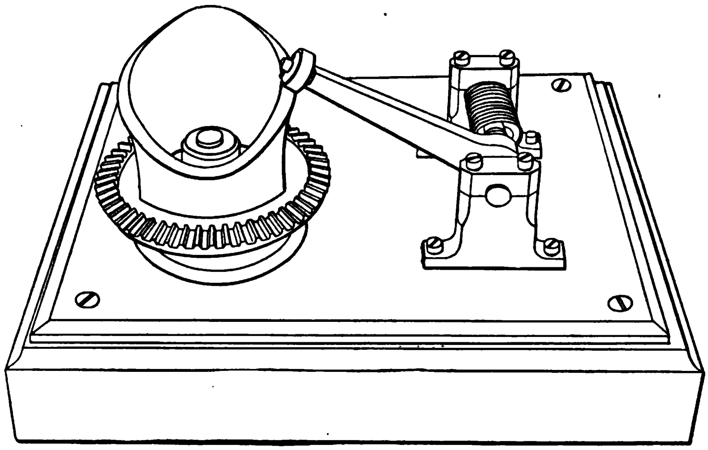

FIG. 149*e*.

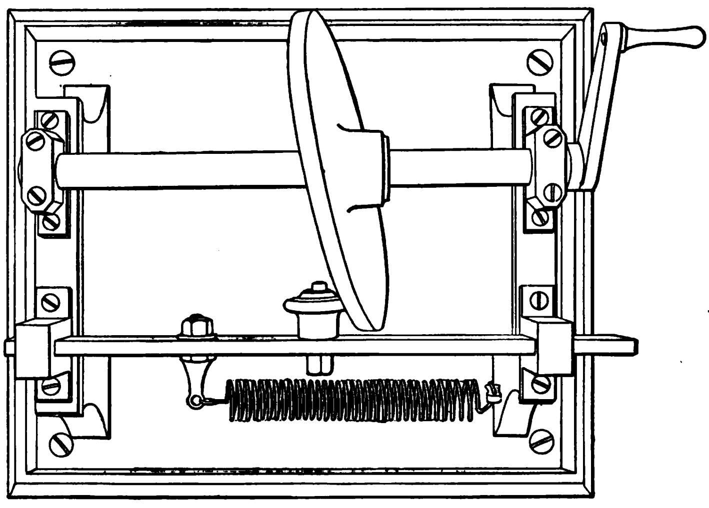

FIG. 149*f*.

In other words, if the cam rotates with uniform velocity, the linear velocity of the follower is proportional to the distance between the centre of rotation of the cam and the virtual

centre of the cam and follower. If the point P moves in an arc of a circle about a point O_{ab} as in Fig. 149*b*, then it may be shown, exactly as in § 63, that the angular velocity ratio of b and c will be the number

$$\frac{\overline{O_{ca}O_{bc}}}{\overline{O_{ab}O_{bc}}}.$$

A number of the forms taken by cam-trains are illustrated by the models represented in Figs. 149*a*–149*f*. A positive-motion cam, of form somewhat similar to that of Fig. 142, is shown in Fig. 149*a*, while Fig. 149*b* is an example of a cycloidal cam whose follower point moves in a circular arc. Fig. 149*c* shows another positive-motion cam, where the follower-pin works in a groove in the cam, and Fig. 149*d* is a sliding cam. In Fig. 149*e* we have a rotating globoidal cam actuating a lever; Fig, 149*f* represents the form of cylindrical cam known as a "swash-plate." The reader will notice in three of these cases the springs which close the pair.

CHAPTER IX.

RATCHET MECHANISMS AND ESCAPEMENTS.

75. **Ratchet-gearing.**—We have so far considered mechanisms in which relative motion of the various links is possible at any instant, so that no link is definitely held or checked by another. We have now to study the action of *Ratchet-gearing*, which may be said to be gearing so arranged that certain links are temporarily or periodically locked together or connected during the action of the mechanism. This locking or checking of relative motion may be so effected that relative motion of the two links is only possible in one sense or direction (when the gear is called by Reuleaux a *Running-ratchet Train*), or movement in both directions may be rendered impossible when the ratchet acts, in which case the gear is known as a *Stationary-ratchet Train*. Fig. 150 shows the two kinds of ratchet-train in their typical

Running. FIG. 150. Stationary.

forms. Each consists of a frame or arm a, a ratchet-wheel b, and a ratchet or click c. In the first figure b is evidently capable of left-handed rotation only, so long as the ratchet c (sometimes called a *pawl*) is resting against its teeth. In the second figure motion is only possible when the pawl is

lifted clear. Examples of simple ratchet-trains will readily occur to the reader; in Fig. 151, for instance, is shown the

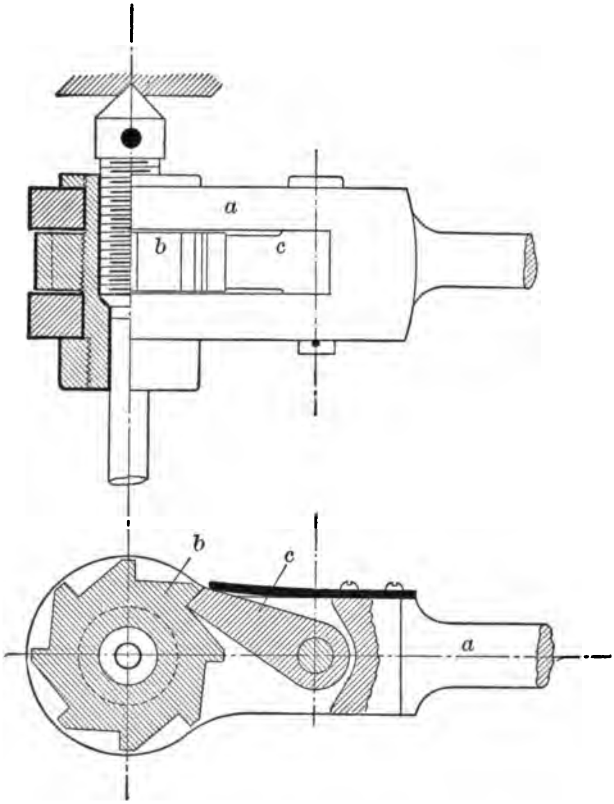

FIG. 151.

mechanism of a ratchet-drill, in which the different links are lettered in the same way as in the preceding figure.

76. Running Ratchets.—It is not necessary that the connection between the pawl and ratchet-wheel in a running ratchet should be of the positive kind shown above. Fig. 152 shows a form of frictional ratchet gear commonly used to transmit motion in one sense only from the crank-axle to the sprocket-wheel of a "free-wheel" bicycle. Here the ratchets themselves, *cc*, take the form of small rollers held up

by springs behind them; the rollers are confined within a driving-ring, *b*, attached to the sprocket-wheel, and when in action jam between this ring and suitably formed surfaces on a ratchet-wheel, *a*, attached to the crank-axle. Such

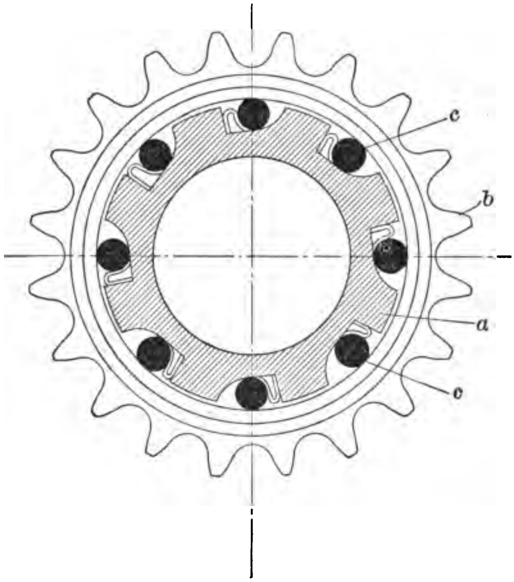

FIG. 152.

frictional ratchet gears are sometimes classed under the head of *silent ratchets*.

It should be noted that while ratchet-trains are used most frequently for controlling the motion of a turning pair, there are many cases in which such trains actuate links which have linear motion.

Fig. 153 shows a running-ratchet gear in which the ratchet *c*, attached to a reciprocating bar *d*, acts on a ratchet-rack *b*, and drives it in one direction only, motion in the opposite direction being prevented by a second ratchet or pawl *c′*, attached to the fixed link *a*. The mechanism is thus a combination of two running-ratchet trains, *abcd* and *abc′*; the former for driving, the latter for checking.

Most running ratchets in common use are really a combination of this kind; for example, in the ratchet-drill the function of the checking ratchet is performed by the frictional resistance of the drill in its hole.

It is important to note that the form of the surfaces on which the pawl and ratchet-wheel or rack engage must be carefully chosen, in order that the mechanism may fulfil its purpose. The shape of the pawl must in fact be such that the pressure between it and the tooth or surface with which it acts does not tend to throw it out of gear. Further, the mechanism must be force-closed, so that the pawl always tends to engage itself; this is commonly effected either by the action of springs (Figs. 151 and 152), or by the weight of the pawl itself (Figs. 150 and 153), or, in some cases, by making the pawl itself a spring. Fig. 154 shows a running friction ratchet which depends for its action on the weight of the ring-shaped pawl itself. Such a mechanism has been employed in certain electric arc lamps for controlling the downward movement of the carbons.

77. Stationary (Checking and Releasing) Ratchets.—Ratchet mechanisms of this type are used where it is necessary to check and release the driven link at will. In most cases a running ratchet or a cam is provided for the purpose of actuating the link whose motion is controlled by the locking ratchet. The mechanism of a *lever-lock* (shown diagrammatically in Fig. 155) is of this kind. The tumbler c and the bolt b here form a stationary-ratchet mechanism with the frame a.

The release of the bolt is effected by the action of the portion M of the key, which really forms a cam engaging with the curved surface of the profile PQ of the tumbler. When this release has been effected the bolt is shot back by the action of the portion N of the key. This part (also a cam) moves the bolt by engaging with the notch seen on the under side of the bolt. In actual lever-locks three, four, or more tumblers are used, with a corresponding number of steps on

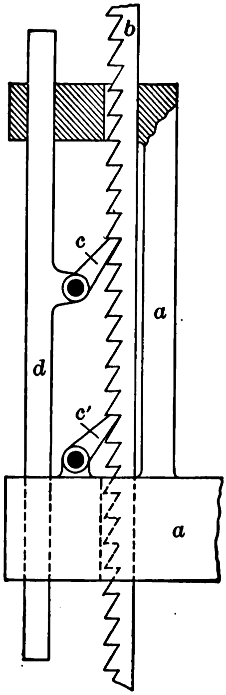

FIG. 153.

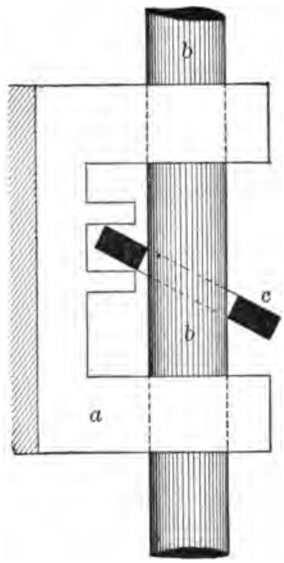

FIG. 154.

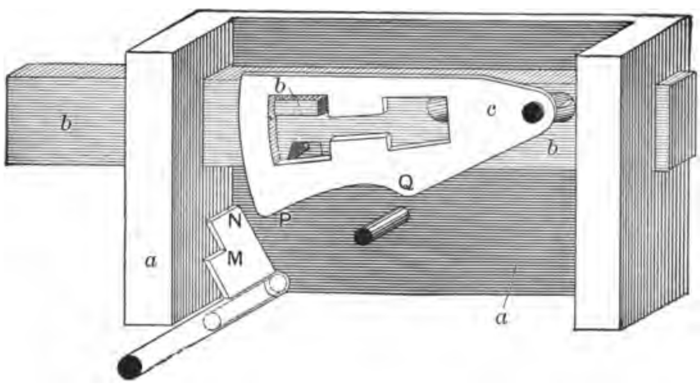

FIG. 155.

the key, and springs are provided so as to press the tumblers against the key.

Releasing and checking ratchets need not necessarily be positive in their action; they may depend on frictional forces just as in the case of the driving ratchet of Fig. 152. Thus, for example, a friction-brake may be looked upon as a frictional checking ratchet.

In Figs. 156*a* and 156*b* we have another example of a checking-ratchet train, in the case of the *Yale lock*. This

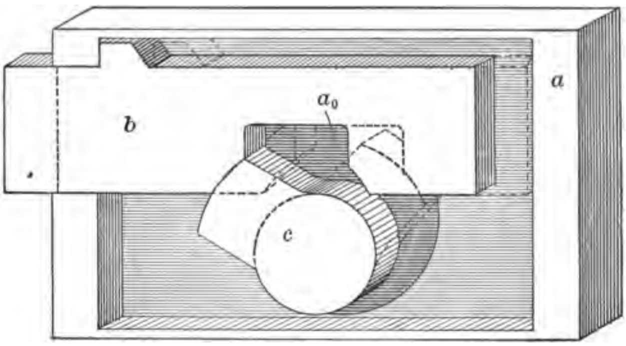

FIG. 156*a*.

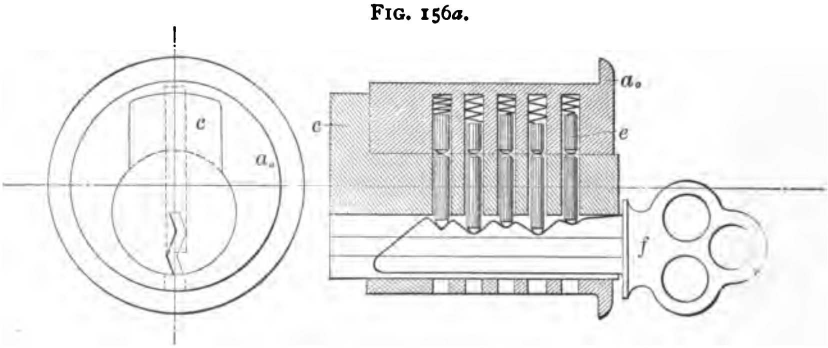

FIG. 156*b*.

lock really contains two distinct mechanisms, one a cam-train *abc*, which actuates the bolt, and the other a locking-ratchet train, which secures the cam, and can only be released by the insertion of the correct form of key. These mechanisms are shown separately. Fig. 156*a* shows the

former train, in which the cam *c* is rotated by turning the key, and locks the bolt when in its extreme outer position. Fig 156*b* shows the cam and its bearing; on inserting the notched key *f*, as shown, each of the tumblers or pawls *e* is lifted to such a height that the division between the two portions of the tumbler is flush with the surface of the bearing. The cam can then be rotated and the bolt *b* can be shot or withdrawn. This locking gear is, of course,

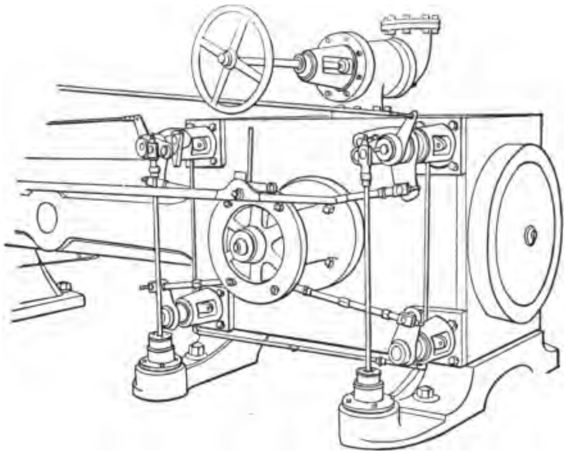

FIG. 157*a*.

a stationary-ratchet train. The case a_0 is rigidly connected with the frame of the lock, *a*, when the whole lock is put together.

Most checking or releasing ratchets are found combined with some form of cam gear, as in the examples above. This is also shown in the case of the releasing-ratchet trains employed for working the steam-valves of a Corliss engine (Figs. 157*a* and 157*b*). Fig. 157*a* represents the engine cylinder and the gear for working its steam- and exhaust-valves; Fig. 157*b* shows in diagrammatic form the ratchet mechanism of the steam-valves. The various parts are arranged somewhat differently in the two figures. The object of such gear is to open the valve at the proper point in the revolution of the engine, and then, after a variable interval, de-

pending on the amount of steam to be admitted, to release the valve so that it may be promptly closed by the action of springs or of gravity. The valve is attached to the spindle and lever *a*, and is opened by rotation in the sense shown by the arrow.

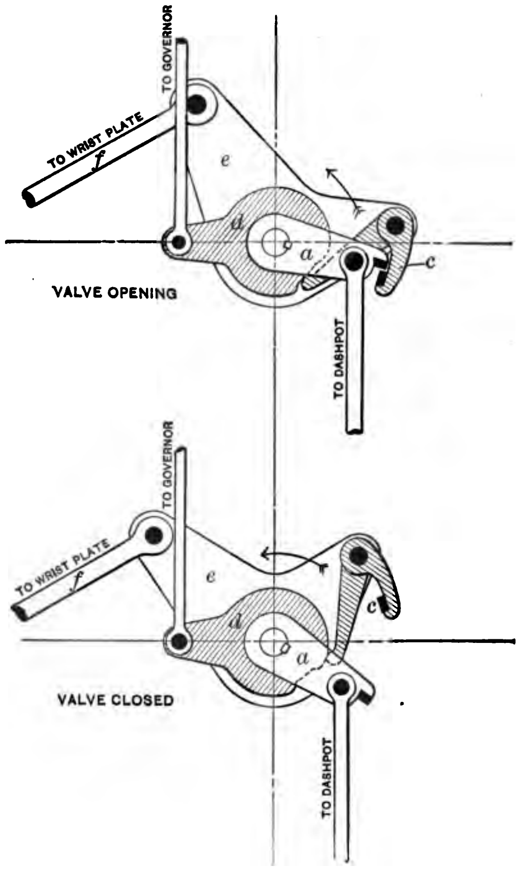

FIG. 157*b*.

In the example shown, the point at which the valve closes is determined by a cam *d* whose position is regulated by the governor of the engine. During the motion of opening, the valve is driven from the rod *f* connected to a rocking wrist-plate (Fig. 157*a*). The wrist-plate thus gives a

rocking motion to the lever *e*, and when moving in the direction of the arrow this lever opens the steam-valve by the engagement of the ratchet or pawl *c* with a corresponding stud or projection on *a*. On reaching the proper point the pawl is lifted by the action of the cam *d*; then the weight of the dashpot, or the tension of a spring, causes the lever *a* to drop. Thus the valve is promptly closed. A spring (not shown) is of course required in order to keep the pawl *c* pressed against the cam *d* and in readiness to engage with *a* on the return stroke.

The many forms of brakes and clutches may be regarded in a sense as ratchet mechanisms (checking and releasing ratchets); in many cases their action is independent of the sense in which the wheel or shaft is rotating.

Fig. 158 shows two forms of clutch employed for connecting at will two pieces of shafting, *A* and *B*. To the shaft *B* is secured one portion of the clutch B_1; the shaft *A* carries

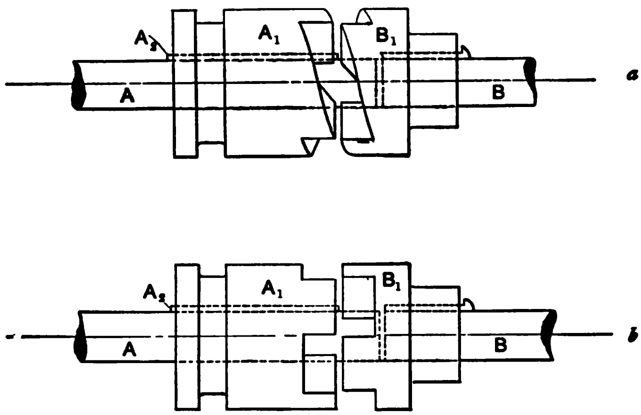

FIG. 158.

the other portion, A_1, in such a fashion that A_1 may be made to slide along *A* so that its projections will engage with the corresponding recesses in B_1. At the same time the projecting feather or key A_2 compels A_1 and *A* to rotate together. Thus when the clutch is engaged, the rotary motion

of B is necessarily transmitted to A. On comparing Figs. 158 and 150 the reader will see at once that we have in the two forms of clutch an exact equivalent of the running and stationary ratchet of § 75. The clutch shown in Fig. 158 in the upper view will only transmit relative motion in one sense; it is therefore really a running-ratchet gear. In the lower view no relative movement of the shafts is possible when the clutch is engaged; the contrivance thus forms a stationary ratchet.

An example of a frictional running ratchet was given in § 76. Fig. 159*a* represents a locomotive wheel and its brake; here we have essentially a frictional stationary-ratchet gear used as a brake, the brake-block corresponding to the ratchet or click. In Fig. 159*b* we have a frictional

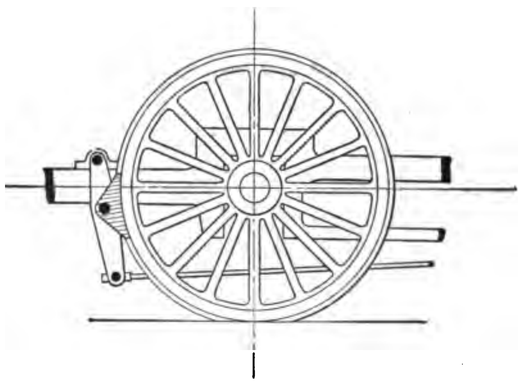

FIG. 159*a*.

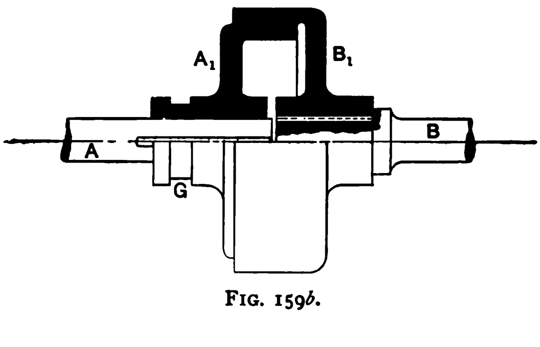

FIG. 159*b*.

stationary ratchet used as a clutch for communicating motion from the shaft B to the shaft A. When the clutch A_1 is pressed along the shaft into contact with B_1 the frictional grip between the two halves of the clutch is sufficient to drive the shaft A. The half clutch A_1 is made to slide along the shaft by the action of a fork whose jaws engage in the groove G shown in the sketch. The same arrangement is employed in the clutches of Fig. 158.

Ratchet mechanisms are of very frequent occurrence in machinery, and it is here impossible to attempt any exhaustive catalogue of their many forms. The subject has been most completely treated by Reuleaux.* Certain ratchet mechanisms containing non-rigid links are discussed in § 88.

78. Escapements (Uniform, Periodical, and Variable).—Under the head of *escapements* may be classed a number of self-acting checking and releasing ratchet mechanisms in which the driven link is alternately released and stopped. The most familiar example is, of course, found in a clock or watch, where the driving weight or spring is permitted to move the clock-work and the hands by a definite amount at regular intervals. Such an escapement is a *uniform escapement*. A second kind of escapement (e.g., the striking mechanism of a clock) allows a train of wheel-work to move at definite intervals, but the amount or range of movement is varied in a predetermined manner, so that, for instance, at every hour the striking gear makes one more stroke than at the preceding hour, up to twelve strokes, after which the cycle commences again. We have here a *periodical escapement* in which, while the period is constant, the range is periodically variable. There is still a third kind, an *adjustable escapement*, in which the range or the period is variable at will or is altered irregularly. We shall take an example of each kind.

* The Constructor, Chapter XVIII; Kinematics of Machinery, §§ 119–121.

Graham's Escapement (Fig. 160) belongs to the first class, and its essential parts are an escape-wheel a (connected with the wheel-work of the clock and driven by it in the sense

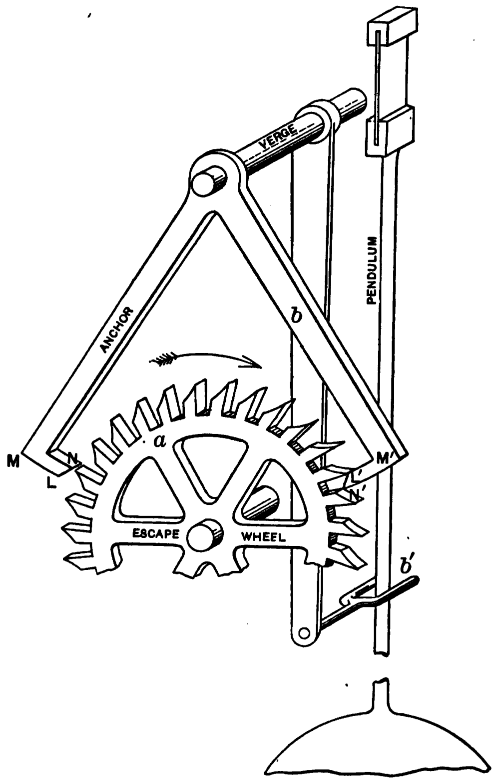

FIG. 160.

shown by the arrow), and an anchor b, whose motion is controlled by the pendulum, with which it is connected through the verge and fork b'. The escapement must (1) permit the escape-wheel to advance by one tooth at each swing of the pendulum, and must also (2) communicate at each swing a minute impulse to the pendulum, so as to maintain its periodic motion. The anchor is really a ratchet,

the surfaces LM, $L'M'$ forming the working faces of the pawl when the motion of the escape-wheel is checked. The faces LN, $L'N'$ are slightly inclined to the circle passing through the tips of the teeth of the escape-wheel, so that as each tooth is driven past the pallet or point of the anchor, a small impulse is given to the pendulum while near the centre of its swing. Almost immediately after a certain tooth has passed LN, for instance, the anchor swings from right to left, and the escape-wheel is checked, because another tooth strikes the face $L'M'$ only to be released when the pendulum again swings back. The curved portions of the tooth-outlines are so formed as to clear the points of the pallets while the anchor is receiving its impulse. It is important that a good clock escapement should work well with a very small angular movement of the pendulum; and in this respect Graham's escapement was a great advance on its predecessors.

As an example of a *periodical escapement* we may take the so-called "English" striking-train of a clock. Fig. 161 shows this mechanism in a diagrammatic form, omitting all unnecessary details. It is desired to communicate to the hammer of a bell or gong such a periodic motion that at stated intervals, say of one hour, the bell is struck; the number of strokes increasing by one each time the movement occurs until the cycle is completed. The whole contrivance includes

(1) A train of wheels (c_1gjk) set in motion by its own driving weight or spring and checked by a ratchet which is released every hour by the clock itself.

(2) A mechanism (driven by the clock) which controls the range of movement of the wheel train c_1gjk, and thus varies the number of strokes given by the bell.

The first part of the escapement consists of the wheel c_1, the cam c_0, and the single-toothed wheel c, all rigidly connected; gearing with c_1 is the wheel g, provided with a pin on which the pawl f_0 acts; and gearing with g is the wheel

k, carrying a number of pins which move the hammer of the bell as k rotates. The whole of this gearing is driven in the sense shown by the arrows and is not directly connected with the driving mechanism of the clock itself. Its movement can only take place when the pawl f_0 is dropped so as to clear the pin on g. There is, however, another way of checking the motion of c_1, g, j, and k. Suppose that the pawl f_0 is released and that c_1 moves in the sense of the arrow

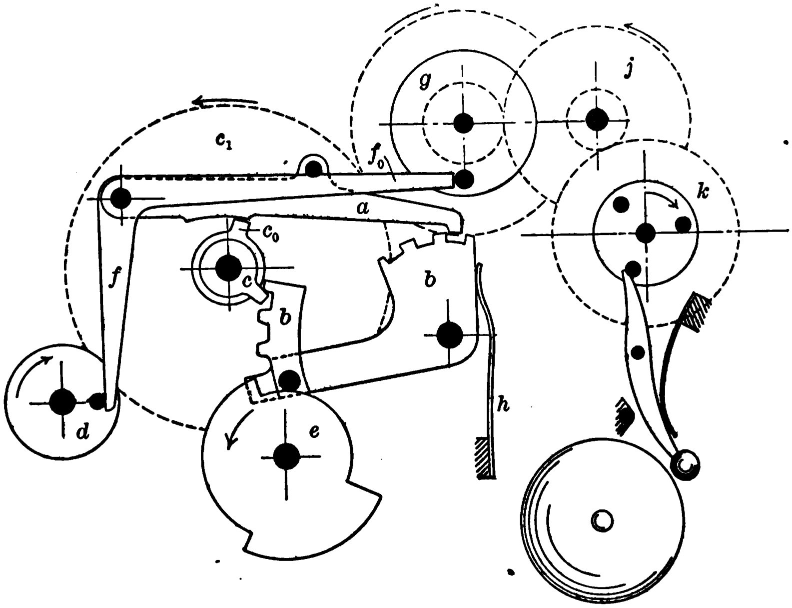

FIG. 161.

from the position shown in the sketch. This motion will continue, and the single tooth of c will engage with the teeth on b until that sector has been lifted to its highest position when c clears the teeth on the sector, and comes in contact with the stop at the lower corner of b. It will be seen that the cam c_0 performs another office, for it lifts the pawl a, and releases b, every time that the tooth on c is in gear, and it also permits this pawl to drop and hold the sector b during the time that the tooth on c is not in gear. The

sector, the wheel and cam, and the pawl thus form a separate locking and releasing ratchet-train and act somewhat after the fashion of the train of § 77.

The upper position of the sector is definite; the lower position evidently depends on the position of the "snail" e. If this snail is driven by the clock in such a fashion that it advances one division every hour, it is evident that the range of the sector b will be altered every hour also. It only remains to arrange that a spring h shall tend to make the sector assume its lowest position, and that a pin on a shall be lifted by f_0 so as to release the pawl a and allow the sector to drop whenever the movement of the train c_1gjk is prevented by f_0.

The action of the whole escapement is then as follows: When the wheel d (which is geared with the snail e) advances beyond the position shown, f_0 drops and permits the train c_1gjk to be set in motion. The bell is then struck as many times as is permitted by the range of the sector b, and that sector is left in its upper position; the motion of b, c, g, and k then ceases. When d has made another revolution and the snail has advanced one division, f_0 is again lifted, the pawl a is also raised, and the sector at once drops, ready for the train to strike again as soon as g is released by the further movement of the wheel d.

In an actual striking-train there will, of course, be twelve divisions on the snail and a corresponding number of pins on k.

The examples here described will serve to give some idea of the nature of escapements of the first two kinds.

Adjustable or variable escapements form a most important class of mechanisms from a practical point of view; they are often of considerable complexity. For instance, the steering-wheel, steering-engine, and rudder of a ship form together a complex variable escapement. This will be understood when it is pointed out that it is necessary for the rudder to move through an angle exactly proportional

to that through which the steering-wheel has been turned; the motion of the steering-engine must then cease and the rudder must be held until the steering-wheel is again moved.

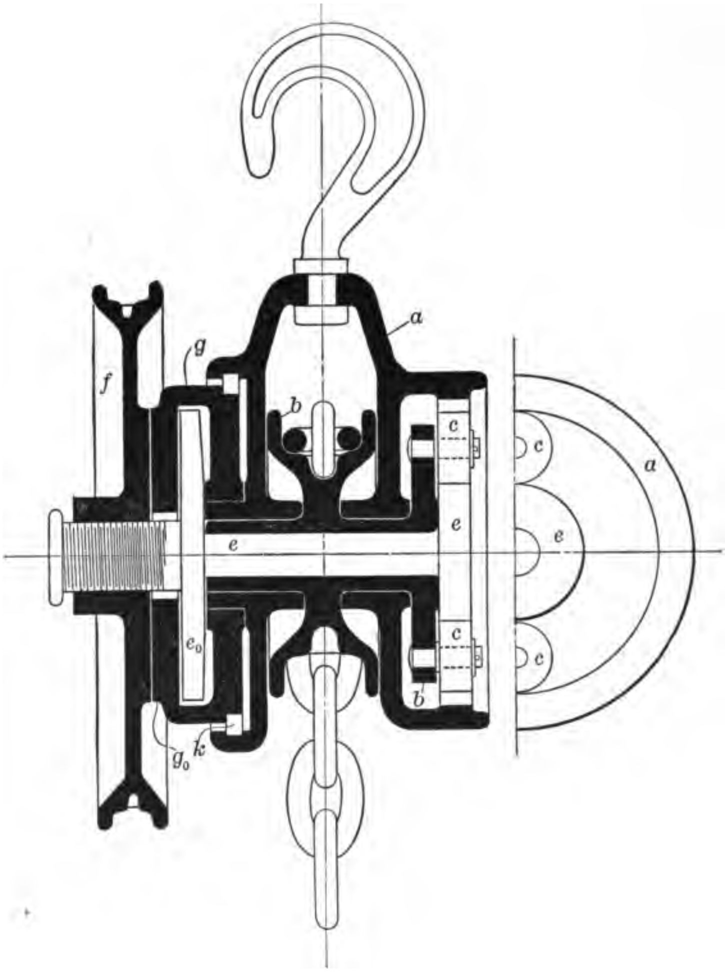

FIG. 162.

Many kinds of lifting and hoisting mechanisms form variable escapements of this type; another well-known example is the hydraulically controlled steam reversing-gear, often applied to large marine engines.*

* See also Fig. 183, § 89.

As an example of an adjustable escapement of a more simple kind, the "Weston Triplex" pulley-block has been selected.* It is shown diagrammatically in section in Fig. 162. The link a forms the body of the block and has on it the bearings of the rotating chain-wheel b, with which the hoisting-chain engages. By means of an epicyclic train *bace* (in which the fixed annular wheel forms part of the link a), b is driven by the rotation of a central shaft e. The hand-chain drives the wheel f, which works on a fine-threaded screw cut on e, in such a way that, when screwed up, a flange g_0 is compressed between the face of f and a corresponding flange e_0 secured rigidly to e. A friction-clutch is thus formed. The flange g_0 forms part of a ratchet-wheel g, connected with a by a roller ratchet like that of Fig. 152. The action of the block may be summarized thus:

(1) Hoisting. The hand-chain wheel f screws up on g_0 and turns the shaft e by means of the friction-clutch fg_0e_0. The ratchet gear *gka* runs freely.

(2) Standing. On ceasing to hoist, the load on the hoisting-chain tends to turn e in the reverse direction; the ratchet gear engages and holds the load.

(3) Lowering. On turning f in the reverse sense, g_0 being held by the ratchet, f is screwed back on its thread, the friction-clutch is released, and the load is lowered *so long as f is kept in motion.* On stopping f the motion of e at once screws up the clutch and checks the load.

The contrivance is thus seen to consist essentially of a rotating shaft driven by an automatic friction-clutch and held by a stationary friction-ratchet, the whole forming an adjustable frictional escapement started and released at will, *the motion of the central shaft imitating that of the hand-chain wheel.* It should be noticed that the action of the machine differs in this important respect from that of a simple hoisting-block provided with an ordinary friction-brake.

* See *Engineering*, August 22, 1890.

CHAPTER X.

MECHANISMS INVOLVING NON-RIGID LINKS.

79. Non-rigid Links.—In giving a definition of a machine or of a mechanism we were careful to use the word "resistant" as applied to the material forming the links composing the mechanism. Many essential portions of actual machines are non-rigid, but are nevertheless resistant, and their occurrence, while it does little to complicate the machine from a kinematic point of view, often introduces dynamical problems of the greatest interest and difficulty. The different classes of non-rigid links, and pairs involving them, have already been noticed; we have now to study certain kinematic questions arising from their use.

In considering non-rigid links in mechanisms or machines it is necessary to take account of the way in which their form changes while in motion. One class of these links is composed of those which, while very yielding as far as bending or thrusting actions are concerned, do not change their length appreciably when a direct pull is applied. Belts, ropes, and chains, which come under this head, are therefore often of great use in machines where energy has to be transmitted in changing directions. This is usually done by causing the flexible tension-links, in the form of belts, ropes, or chains, to pair with, and communicate motion to, rotating drums or wheels. On account of their change of form, non-rigid links can have no virtual axes or virtual centres.

80. Velocity Ratio in Belt-gearing. Length of Belts.—The linear velocity of a rope or belt passing over two or

more pulleys may be considered for kinematic purposes as being the same throughout its length. In practice the stretching of a rope or belt under load often has an appreciable effect on the velocity ratio of the pulley it drives; we shall here treat questions of velocity ratio as if the belt or rope were inextensible. Fig. 163 represents a pair of cylindrical pulleys connected by a belt, which may be "open" or "crossed" so that the pulleys rotate either in the same or in opposite senses. We shall for the present neglect the effect of the thickness of the belt or rope.

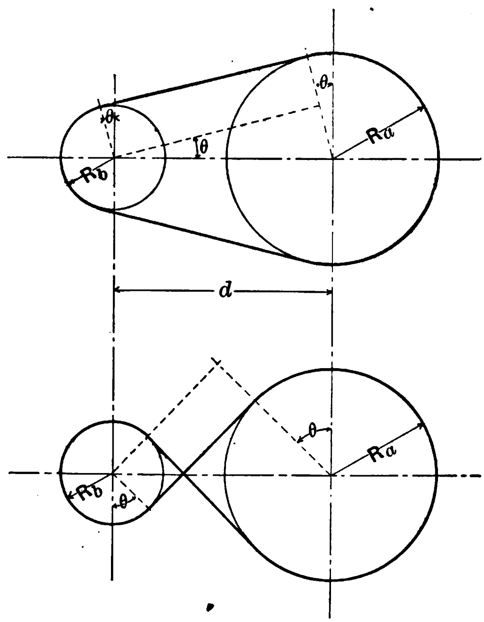

FIG. 163.

In these cases if V be the linear velocity of the belt and R_aR_b the radii of the pulleys, the angular velocity ratio will evidently be found from the relation

$$\frac{\omega_a}{\omega_b}=\frac{V}{R_a}\cdot\frac{R_b}{V}=\pm\frac{R_b}{R_a},$$

the negative sign corresponding to the case of a crossed belt. It is, of course, assumed that there is no slipping.

The length of a belt is easily expressed in terms of the radii and the distance d between the centres of the pulleys. The total length of belt not in contact with the pulleys is

$$2\sqrt{d^2-(R_a \pm R_b)^2},$$

the negative sign here corresponding to the case of an open belt. If θ be the angle that the straight part of the belt makes with the centre line of the pulleys, then the length of belt in contact with the pulleys will be

$$(\pi+2\theta)(R_a+R_b) \text{ for a crossed belt}$$

and

$$(\pi+2\theta)R_a+(\pi-2\theta)R_b,$$

or

$$\pi(R_a+R_b)+2\theta(R_a-R_b) \text{ for an open belt,}$$

where $\theta = \sin^{-1}\dfrac{R_a \pm R_b}{d}$.

The expression for the total length of belt will then be for an open belt

$$2\sqrt{d^2-(R_a-R_b)^2}+\pi(R_a+R_b)+2(R_a-R_b)\sin^{-1}\frac{R_a-R_b}{d},$$

and for a crossed belt

$$2\sqrt{d^2-(R_a+R_b)^2}+(R_a+R_b)\left(\pi+2\sin^{-1}\frac{R_a+R_b}{d}\right).$$

It will be seen that the length of a crossed belt is thus constant so long as the sum of the radii and the distance between the centres of pulleys are constant quantities.

81. Belt-gearing for Variable Velocity Ratio.—Fig. 164 shows the arrangement of "cone pulleys" employed in driving machinery so as to render it possible, by shifting a

belt from one pair of steps to another, to obtain at will any one of several velocity ratios. It is plain that the same *crossed* belt will run with the same tightness on any pair of steps so long as the sum of the radii of each pair is the same. An open belt, however, is generally required, in which case the tension will be different on each pair of steps, unless

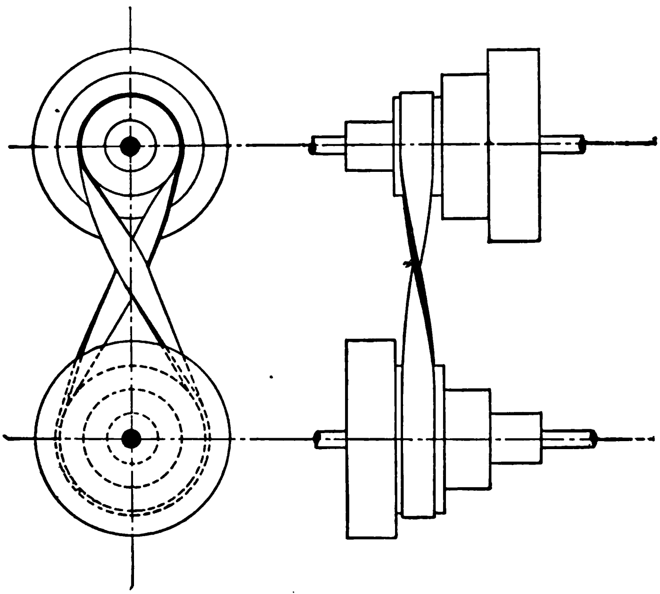

FIG. 164.

their diameters are specially calculated. Approximate methods for readily doing this have been devised,* while Reuleaux † gives a rigorous graphical treatment of the problem. Referring to Fig. 163, we have as an expression for the length of an open belt

$$l=2\left[d \cos \theta+\frac{\pi}{2}(R_a+R_b)+\theta(R_a-R_b)\right].$$

* Unwin, Machine Design, Vol. I, p. 373; Smith, Trans. Am. Soc. M. E., Vol. X, p. 269.

† Reuleaux, The Constructor. Trans. by Suplee, p. 189.

Now $R_a - R_b = d \sin\theta$; therefore

$$l = 2\left(d\cos\theta + \frac{\pi}{2}(2R_a - d\sin\theta) + \theta d\sin\theta\right)$$

and

$$R_a = \frac{l}{2\pi} - \frac{d}{\pi}(\cos\theta + \theta\sin\theta) + \frac{d\sin\theta}{2}.$$

Similarly,

$$R_b = \frac{l}{2\pi} - \frac{d}{\pi}(\cos\theta + \theta\sin\theta) - \frac{d\sin\theta}{2}.$$

Take a pair of rectangular axes OA and OB, (Fig. 165) and make $OA = d$. Draw a curve CD, the involute of the circular arc AC, having O as its centre. Then, since the

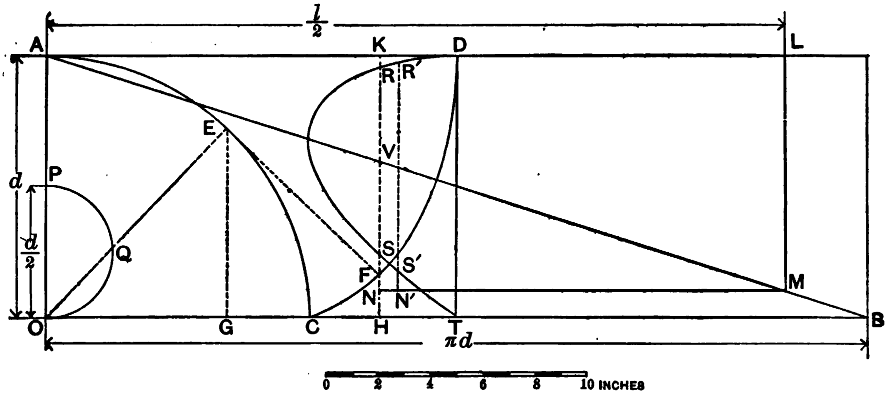

FIG. 165.

angle θ must lie between 0° and 90°, it must have some such value as COE, in which case the line EF, tangent to AC at E, and cutting the involute at F, has a length equal to the arc EC. Hence $EF = \theta d$, and, drawing KFH parallel to AO, we have $GH = EF \sin\theta$, and

$$OH = OG + GH$$
$$= d(\cos\theta + \theta\sin\theta).$$

Next make $OB = \pi d$ and join AB. Draw AD parallel to OB. Let HF meet AB in V and AD in K; then

$$\frac{KV}{OH} = \frac{AO}{OB} = \frac{1}{\pi},$$

and therefore $$KV=\frac{d}{\pi}(\cos\theta+\theta\sin\theta).$$

Again, if we set off $AL=\frac{l}{2}$, and draw LM parallel to AO, and cutting AB in M, we have

$$LM=\frac{l}{2\pi}.$$

Draw MN parallel to BO and cutting HK in N, then

$$\begin{aligned}VN&=KN-KV\\&=\frac{l}{2\pi}-\frac{d}{\pi}(\cos\theta+\theta\sin\theta).\end{aligned}$$

To obtain the value of $\frac{d\sin\theta}{2}$ we need only draw a semicircle OQP having a diameter $\frac{d}{2}$; then

$$OQ=\frac{d\sin\theta}{2}.$$

Finally a curve $DRST$ may be drawn by setting off $VR=VS=OQ$, and repeating the construction as required. This gives

$$\begin{aligned}NR&=VN+VR\\&=\frac{l}{2\pi}-\frac{d}{\pi}(\cos\theta+\theta\sin\theta)+\frac{d\sin\theta}{2}\\&=R_a,\end{aligned}$$

and
$$\begin{aligned}NS&=VN-VS\\&=\frac{l}{2\pi}-\frac{d}{\pi}(\cos\theta+\theta\sin\theta)-\frac{d\sin t}{2}\\&=R_b.\end{aligned}$$

Thus $$R_a-R_b=VR+VS=SR.$$

Plainly for given values of l and d we can determine R_a and R_b for any value of θ (or for any required velocity ratio) by the aid of the curve $DRST$.

In practice it is usual to find that the diameters of the

first pair of steps or their radii, R_a and R_b, are given, together with d, the distance between centres of pulleys. The problem then is to find the radii of another pair of pulleys, R_a' and R_b', on which an open belt of the same length will run with a given velocity ratio $\frac{R_a'}{R_b'}$. The author has found the following a convenient method of utilizing the Reuleaux diagram for solving this problem, and for finding incidentally the length of belt required. This length, however, is not often necessary, as it is more easily measured from the pulleys when finished and in position.

Draw the rectangle AOB and the curve $DRST$ exactly as described above, and as shown in Fig. 165, making OA, say, 10 inches in length. This diagram can be used for finding pairs of radii of steps having any desired velocity ratio, and the lengths of these radii will be obtained in terms of d, the distance between the shaft centres. Having expressed R_a and R_b, the given pair of radii, in terms of d, it is easy, by applying a scale of inches and hundredths to the diagram, to determine that position of the line SR which will give the proper value to $R_a - R_b$. The length RN is then measured to the proper scale and the point N found. If required, the half length of the belt is then settled by drawing the line NM, and the next thing is to find another set of points R', S', N' such that $R'N'$ and $S'N'$ will have the ratio required for the radii of the next pair of steps. This is readily done by drawing on tracing-paper a set of radiating lines (Fig. 165*a*), VX, VY', VZ, arranged so as to cut all lines perpendicular to VZ in the required ratio, namely, $\frac{R'_a}{R'_b}$. It is convenient to draw another line, VY, such that lines perpendicular to VZ are also cut in

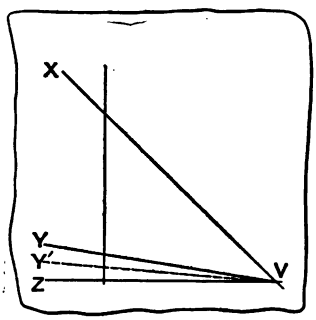

FIG. 165*a*.

the ratio $\frac{R_a}{R_b}$. By applying this diagram to Fig. 165, the three points R', S', N' can readily be pricked off in their proper positions. When measured to the proper scale, $R'N'$ and $S'N'$ give the values of the pair of radii required. In Fig. 165 the ratio $\frac{R_a'}{R_b'}$ is 12.0, while

$$\frac{R_a}{R_b} \text{ is } 6.0 \quad \text{and} \quad l = 5.66d.$$

If the real value of d is taken as 30 inches, while R_a and R_b are 25.2 and 4.2 inches respectively, the diagram gives for R_a' and R_b' the values 26.4 and 2.2 inches. An open belt of about 170 inches in length would run on either of these two pairs of pulleys.

It should be noted that when d is large in comparison with the size of the step pulleys, it is often sufficiently accurate to proportion the latter as if intended to run with a crossed belt; for this purpose the sum of the radii may be made constant.

To make allowance for the effect of the thickness of the belt or rope in our calculations it is only necessary to reflect that we have really taken the thickness of belt as being negligible when compared to the diameter of the pulley. In practice this is frequently not the case. Suppose, for example, that a belt whose thickness is a quarter of an inch is running on a pulley 6 inches in diameter. We assume that while passing round the pulley the layer of material at the centre of the thickness of the belt is neither stretched nor shortened, so that the arrangement will be equivalent kinematically to a pulley 6¼ inches in diameter on which a belt of negligible thickness is running. In other words, we take it that the effective radius of the pulley is in all cases to be measured to the centre of the thickness of the belt or rope.

82. Velocity Ratio in Chain- and Rope-gearing.—Rope- and chain-gearing is extensively used for the transmis-

sion of power, as well as in machinery for hoisting, winding, and lowering. In many cases it is necessary to provide the rope-drums or pulleys with guiding or retaining grooves. The various forms of rope and chain tackle are too familiar to require extended notice here; the ratio of the speed of the rope to the speed of the body moved by the tackle can always be readily found. As an example, we may take the Differential Pulley-block of Fig. 166. In this case the upper block has two sheaves a and a' rigidly connected or made in one piece; the chain is prevented from slipping on these sheaves by suitable projections in their grooves. Evidently on hauling in the sense shown by the arrow, the loop or bight of the chain passing around b will be shortened during each revolution of a and a' by an amount equal to the difference of the circumferences of those pulleys. Hence, if we call R_1 and R_2 the effective radii of a and a' we shall have

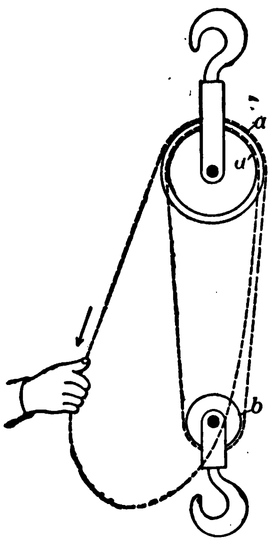

FIG. 166.

$$\frac{\text{speed of chain}}{\text{speed of hoisting}} = \frac{2\pi R_1}{\frac{1}{2}(2\pi R_1 - 2\pi R_2)}$$
$$= \frac{2R_1}{R_1 - R_2}.$$

Sometimes it is desirable to arrange hoisting gear in such a way that the velocity ratio is variable. For instance, in the winding gear of a deep mine it is necessary to wind the rope on a drum of continually increasing radius provided with a spiral groove, so that when one cage is at the bottom, its weight together with that of the attached rope may be balanced by the smaller weight of the other cage alone acting on a portion of the drum which is of larger radius.

A similar device is employed in the "fusee" of a chronometer.

In some cases shafts and pulleys are so connected by chain-gear that their velocity ratio is not uniform throughout the revolution. Fig. 167 shows one form of sprocket-wheel and chain. The wheel is furnished with teeth engaging with the links of the chain and effectually preventing slipping; these teeth should evidently have profiles composed of circular arcs parallel to the paths described by the centres of the pins as they move relatively to the wheel. On considering a pair of such wheels connected by a chain it will be seen that if their pitch-circles are of unequal diameters, their velocity

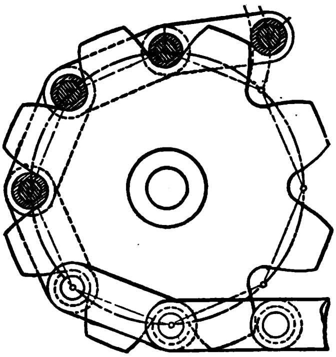

FIG. 167.

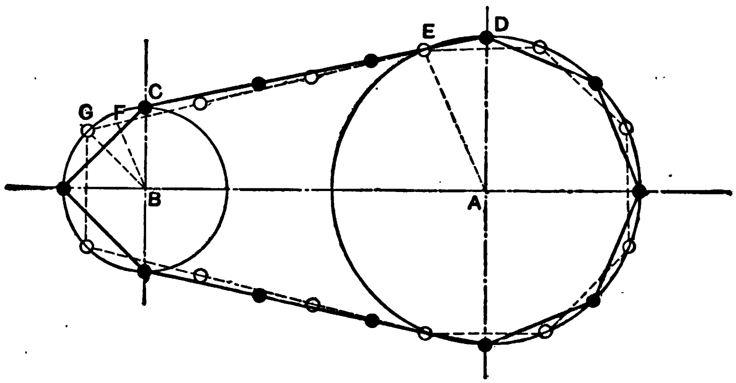

FIG. 167a.

ratio will not be the same in every position. Fig. 167*a* represents the centre line of a chain connecting a pair of sprocket-wheels; the wheels have four and eight teeth respectively. When in the position shown by full lines, the pair of wheels and the chain are equivalent to a four-bar mechanism or quadric crank chain *ABCD*. Applying the

construction of § 28 we find that the velocity ratio is

$$\frac{\omega_a}{\omega_b}=\frac{BC}{AD}.$$

But when in the position shown by dotted lines the velocity ratio is

$$\frac{\omega_a}{\omega_b}=\frac{FB}{AE}<\frac{BC}{AD},$$

the equivalent position of the quadric crank chain being *AEGB*. In sprocket- and chain-gearing as used for cycles this inequality of velocity ratio may amount to from 5 to 15 per cent.

A form of chain used by Mr. Hans Renold for transmitting power between two parallel shafts is shown in Fig. 168. The

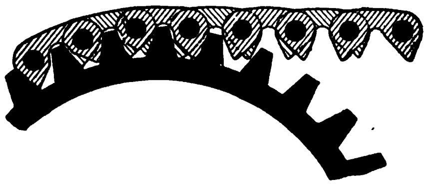

FIG. 168.

chain links have projections or teeth on their inner edges, so formed as to gear with teeth on the wheel rims. It will be seen from the diagram that these teeth profiles, the working portions of which are made up of straight lines, are so arranged that the links enter and clear the wheel teeth without appreciable sliding or rubbing motion. The angle embraced by the sides of the wheel-tooth profile is smaller the smaller the number of teeth in the wheel. Chains of this kind will work correctly even if slight stretching has taken place. The periodical inequality of velocity ratio when the driving and driven wheels are of different sizes is to be determined for these chains in exactly the same way as for ordinary pitch chains, the effective diameters of the wheels being, of course, measured to the centres of the pins of the chain links.

83. Belt- and Rope-gearing between Non-parallel Axes.— It should be noted that belts and ropes may be used for

transmitting power between shafts whose axes are not parallel; in some cases idle guide-pulleys are required in order that the belt or rope may run satisfactorily. For this to be the case one condition must be fulfilled, namely, that wherever a belt or rope is running on to a pulley the centre line of the advancing belt or rope must lie in the central plane of the pulley on to which it is running; i.e., in a plane normal to the axis and passing through the centre of the pulley.* A number of cases of belt transmission between non-parallel axes are illustrated here. The belt in Fig. 169*a* can only

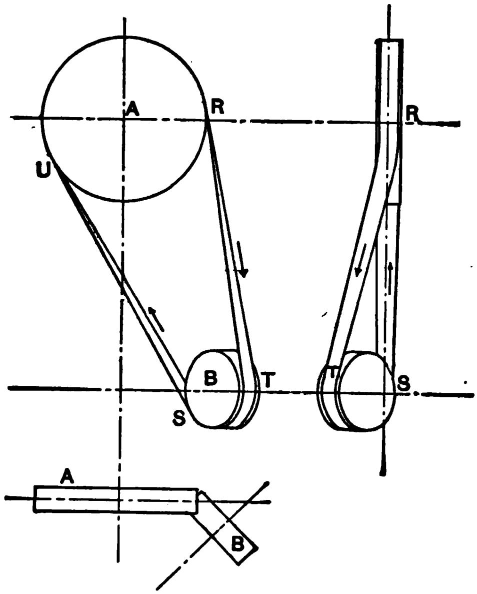

FIG. 169*a*.

be run in the direction indicated by the arrows, the portion *RT* lying in the plane of the pulley *B*, but not in that of *A*, while the part *SU* lies in the plane of *A*. Similar remarks apply to Fig. 169*b*, where, however, a guide-pulley is used. In Fig. 169*c*, it will be seen that if the guide-pulley *C* is placed in a plane containing the parts of the belt *S T* and *UT* every straight portion of the belt lies in a line which forms

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* See Webb, Trans. Am. Soc. M. E., 1883, p. 165.

the intersection of the planes of its pulleys; hence the belt will run either way. Fig. 169*d* shows the general case when the axes are inclined. Any two points, X and Y, are chosen on the line forming the intersection of the planes of the pulleys A and B, and tangents XR, XS, YT, YU are drawn to these pulleys. The guide-pulleys C_1 and C_2 are then

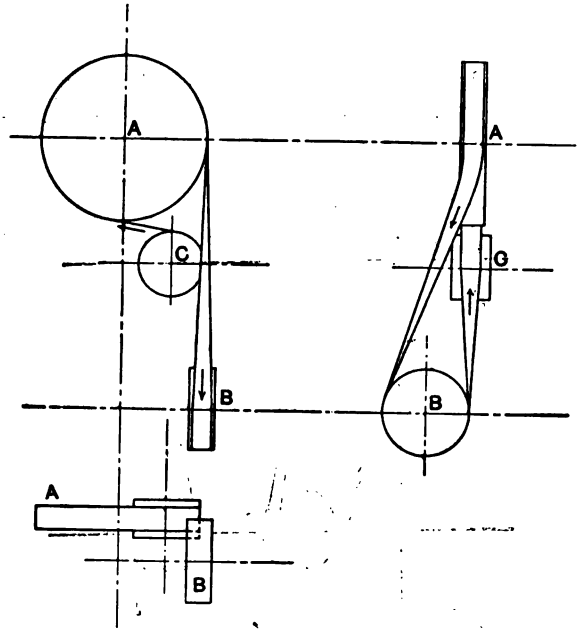

FIG. 169*b*.

placed in the planes XRS, YTU' respectively. Under these conditions the belt will run either way. These examples will serve to indicate the method to be adopted in arranging belt-gearing when the axes of the shafts are not parallel.

Similar remarks apply to arrangements for rope-gearing, but in this case, as the pulleys are grooved, guide-pulleys are not so frequently required.

84. Springs.—While belts, ropes, and chains are especially of use for transmitting energy, the flexible links known as *springs* are of service where energy has to be stored up and again restored when required. Problems connected

with mechanisms involving springs will in general deal with questions of Dynamics rather than with questions of Kinematics; it will be sufficient here to notice some cases in which the energy of springs is employed for kinematic purposes; i.e., for controlling or assisting the relative motions of machine parts.

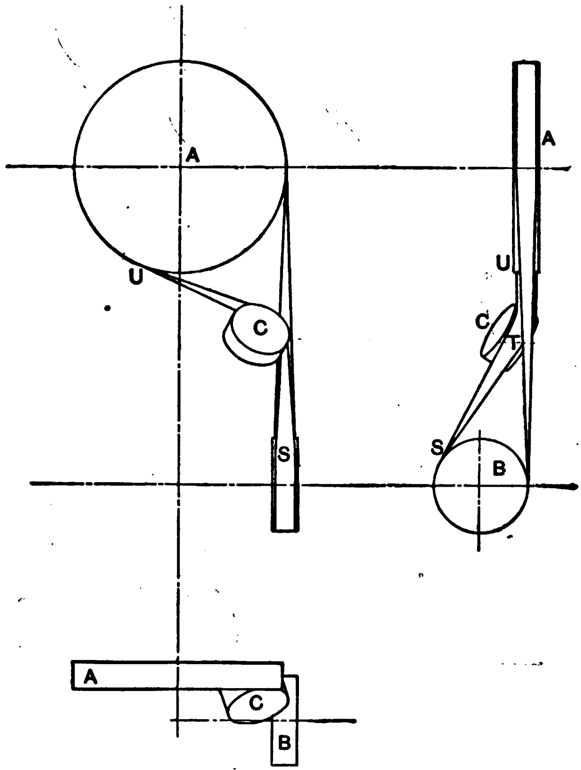

FIG. 169c.

Springs are often used for the closure of mechanisms and pairs. The spring *h*, for example, in Fig. 161 supplies the force required to keep *b* in contact with *e* or in contact with *c*, as the case may be. Certain springs in gun-locks fulfil a similar purpose, and springs of the same kind form an essential feature in most ratchet mechanisms and escapements.

In Fig. 170 the essential parts of the lock of a Winchester rifle are shown. The lock mechanism contains two springs; the main-spring *a* is bent when the hammer *b* is drawn back

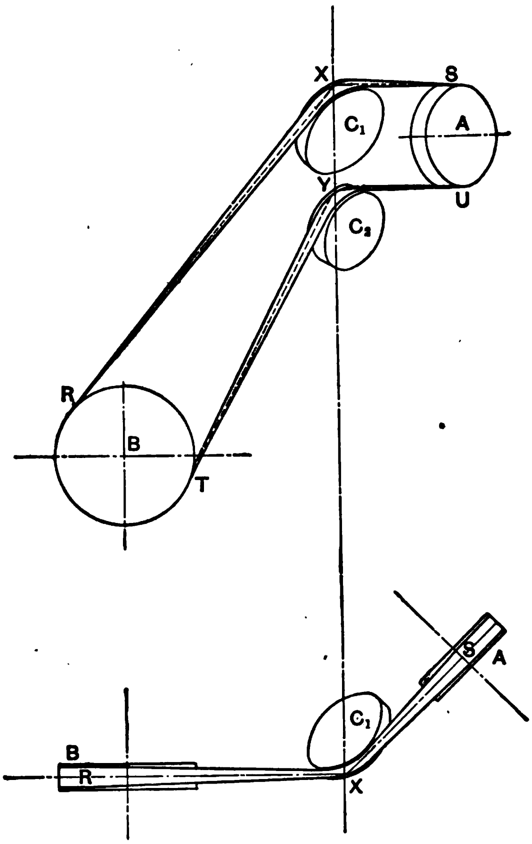

FIG. 169*d*.

and cocked, and the energy stored in this spring is available, when released, for striking the firing-pin *c* and thus exploding the cartridge. The trigger-spring *d* bears on the trigger *e*, which serves in fact as a pawl or detent for the hammer. When the hammer is at half-cock the point or nose of the trigger enters the first notch on the hammer; the hammer is then secure, as the form of the notch prevents the trigger from being pulled. When

the hammer is placed at full cock, however, the point of the trigger engages with the second notch, which is of such a form that the trigger can be pulled and the hammer released. While both these springs may be regarded as serving for purposes of closure, *d* has no other use; *a*, on the other hand, stores up energy in the way already described. The whole mechanism forms a locking and releasing ratchet-train (see § 77) which is spring-closed.

In many cases springs are used simply as means of storing energy, a very familiar example being the coiled spring

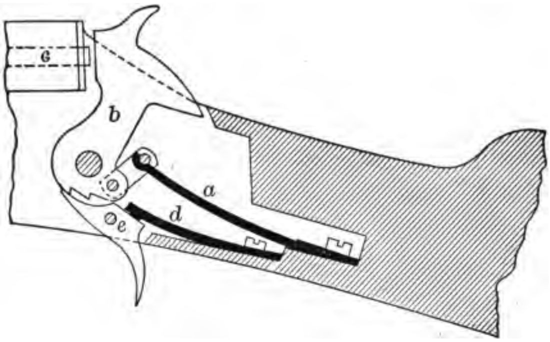

FIG. 170.

which drives a clock or watch; in other instances they are employed simply to control or modify the relative motion of machine parts. The springs in a buffer, or the springs which hold down a safety-valve, come under this heading.

85. Fluid Links. Pressure Pairs.—It has already been noted that we class under the name of spring those portions of mechanisms whose elastic deformations, when the mechanism is in action, are considerable as compared with the dimensions of the spring itself, and with the extent of motion of the other links of the mechanism. According to this definition we ought to include in our list not only solid springs, but also such bodies as the air in an air-compressor, which, although fluid, suffers elastic deformation.

Hydraulic machines, again, contain fluid links which do

not sensibly change their volume under the pressures to which they are subjected in working. Hence fluid links may be divided into (*a*) elastic and (*b*) non-elastic links. As has been previously stated, the changes of form and volume of these links involve questions of dynamics which lie outside of the scope of the present work, so that we shall here consider only in their kinematic aspect certain mechanisms containing fluid links.

In every case the pairing of the fluid link with the solid link or links containing it will be "pressure pairing" (see § 9); in using fluid links in mechanisms we therefore meet with a constructive difficulty not found when employing rigid material only, namely, that all moving parts in contact with fluid under pressure have to be so made that no unnecessary leakage can take place. The means of attaining this object is not important from a purely kinematic point of view.

A large number of mechanisms containing fluid links will be found to have as their counterparts mechanisms containing rigid links only. We find, for example, many in-

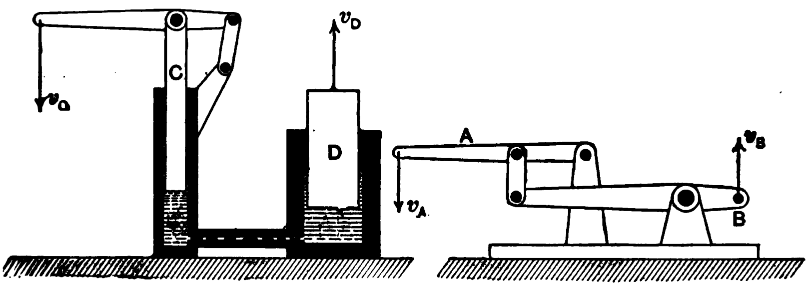

FIG. 171.

stances of fluid ratchet-trains, and Fig. 171 compares the well-known hydraulic press with a system of rigid levers whose ends AB have the same velocity ratio as the plunger C and ram D of the press. By suitably proportioning the areas of C and D we can obtain any desired ratio of the rising velocity of D to the falling speed of C, for it is plain

that the volume of water displaced in a given time by C must be equal to that entering the cylinder in which D works. Hence the speed of D will be to the speed of C inversely as their areas, assuming that the fluid is incompressible and that no leakage occurs.

It is evident that under these circumstances the relative velocity of C and D will be unaffected by the length or form of the pipe or passage communicating between the cylinders. These might in fact be separated by a considerable distance, in which case the mechanism would serve for the hydraulic transmission of energy. Such transmission is found of great utility under certain conditions. Similarly arrangements for the transmission of power by compressed air have been devised; in either case it is the fluid link which renders this type of transmission possible and economical.

86. Chamber Crank-trains.—The most important application of the crank-chain in machine construction is its use—in kinematic combination with a fluid link—for the purpose of an "engine" or prime mover, or for the purpose of a "pump" or machine for moving or compressing the fluid. The fluid link may consist of steam, air, gas, or water, and the mechanism must include a suitable chamber for enclosing it. We proceed to give a few examples of such *chamber crank-trains*, selected from the numberless instances of every-day occurrence.

Any of the inversions of the slider-crank chain of Chapter IV may be converted into a chamber crank-train if we make one of its links into a vessel or chamber and convert another link (in some cases two others) into a plate or diaphragm moving in the vessel in such a fashion that the fluid link occupies the space enclosed. As the mechanism operates, the effective volume available for the fluid link is changed and the fluid expands, or is compressed; it enters or leaves the vessel or chamber in conformity with the alteration in volume and the conditions under which the machine is working. The ordinary direct-acting steam-engine

is a familiar example and is derived from the turning slider-crank of Fig. 60. The link c forms the piston or movable diaphragm, while the link d takes the shape of the cylinder in which the piston travels. In some instances the cylinder or chamber is so formed as to enclose partially or completely the links a and b. Fig. 172 shows diagrammatically how this is done in the case of the small enclosed petroleum or gasoline motors so much used for the propulsion of automobiles and boats. In Fig. 68, again, the *swinging block slider-crank* is used as an oscillating steam-engine; c is now the cylinder and d the piston and rod.

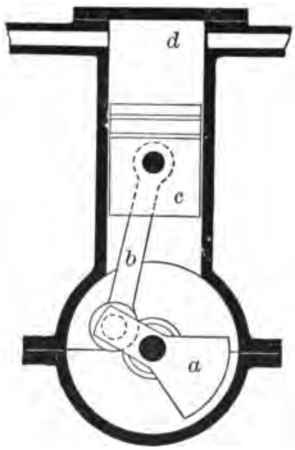

FIG. 172.

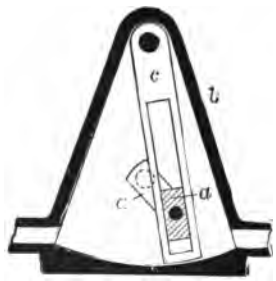

FIG. 173.

It is also possible in the swinging block slider-crank to convert the fixed link into the chamber. Fig. 173 shows a method of doing this, suggested by Reuleaux. On consideration it will be obvious that this arrangement, like many others quite possible kinematically, will not be likely to give satisfactory results in practice. The mechanical difficulties of making the contact between c and b sufficiently good to avoid leakage, and at the same time so free as to avoid frictional loss, are so great that a large number of possible chamber crank-trains are of no practical value.

Passing on to the *turning block slider-crank chain* of Fig. 73, this has been converted into a chamber train, and was originally proposed as a steam-engine by Lord Cochrane in 1831 and 1834. Probably the inventors of so-called "rotary" engines and pumps have nowhere found so extensive a field for their ingenuity as among mechanisms derived from this kinematic chain. One form of the Cochrane engine is shown in Fig. 174. Here the rotating chamber is formed from the link *b* of Fig. 73*a*, and has line contact with the link *d*. The fixed link *a* (corresponding to the crank in a direct-acting engine) forms the frame or support of the mechanism, and the working fluid expands in the spaces enclosed between *b*, *c*, and *d*.

The *swinging slider-crank* when used as a chamber train has already been shown in Fig. 74.

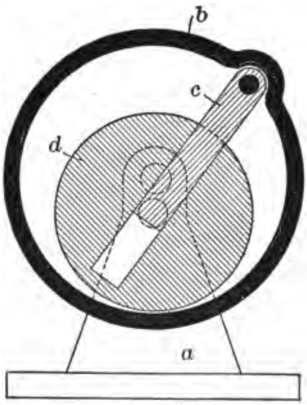

FIG 174.

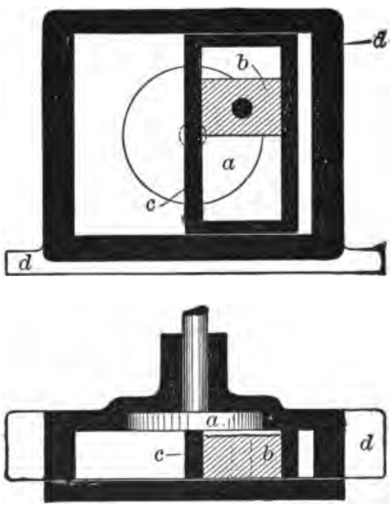

FIG. 175.

From the *double slider-crank chain* a number of chamber trains can be derived. Fig. 79 shows one form—the common donkey-pump. In Fig. 175 we have a form of steam-engine proposed by Root in 1864 and since revived by other inventors. On comparison with Fig. 77 or 79 the corre-

sponding links will be readily recognized. In the Root engine the rectangular spaces enclosed between the links *c* and *d* and between *b* and *c* are used as chambers to receive the working fluid; their varying volume serves for the expansion of the steam. It should be noted that this form of engine does not involve the use of higher pairing, and there is not the difficulty experienced in so many chamber crank-trains in preserving a steam-tight joint. When the surfaces between *a*, *b*, *c*, and *d* are adjusted so closely as to avoid leakage, there is in practice found to exist a considerable amount of friction, and the accuracy of adjustment is easily destroyed by expansion due to any slight local variations in temperature of the different parts of the engine. It is for reasons of this kind that no form of chamber crank-train has yet been able to compete in practice with those types derived from the turning slider-crank.

It is, of course, to be understood that in a chamber crank-train used as a motor or pump suitable provision must be made for the government of the admission and outflow of the working fluid. This is sometimes effected by arranging the necessary openings and passages so that they are opened or closed by the motion of the solid links themselves. More frequently it is necessary to provide a subsidiary ratchet-train or *valve-gear*, which forms no essential portion of the original machine, if we consider only the motion and pairing of the solid links, the object of the valve-gear being simply the control of the fluid link. These mechanisms are considered further in § 89.

87. Chamber Wheel-trains. — Reuleaux divides motor-mechanisms containing "pressure organs" or fluid links into two classes. We have first those mechanisms in which the motion is more or less intermittent, so that the whole machine forms a "fluid ratchet-train." The kinematic chains discussed in the last section, when provided with the necessary valve-gear, belong to this class. The second class includes those "running mechanisms" in which the motion

of all the solid links is continuous, and we now proceed to consider some examples of this kind of chain, formed by modifying certain wheel-trains in such a way as to constitute a *chamber wheel-train*. The chamber is, in general, formed from the frame of the wheel-train and carries the wheels by means of simple turning pairs. The fluid or working substance occupies the space between the wheels and the chamber, and such mechanisms, in spite of certain mechanical disadvantages, are often used as pumps or motors, or as meters for measuring the amount of fluid passing through the chamber.

The chamber wheel-trains which are simplest from a kinematic point of view are those containing only one wheel and the necessary chamber and passages for the guidance of the fluid link. In Fig. 176, for example, is shown diagram-

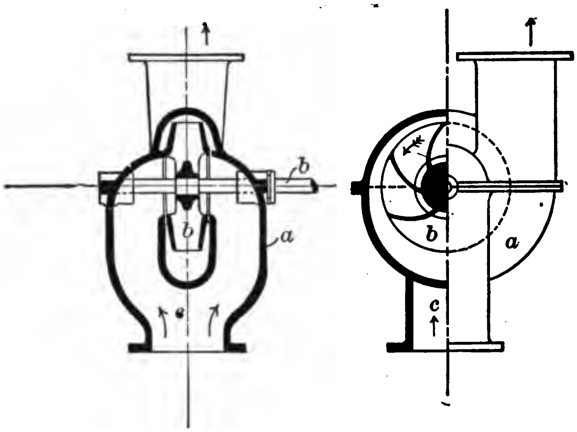

FIG. 176.

matically a centrifugal pump, the whole mechanism consisting of the pump-casing a, the wheel and shaft b, and the fluid c. A turbine, or water-wheel, of course falls into the same class.

Figs., 177a, 177b and 177c show three types of chamber wheel-gear amongst many which have found some degree of favor in practical use as pumps or motors. Fig. 177a is the

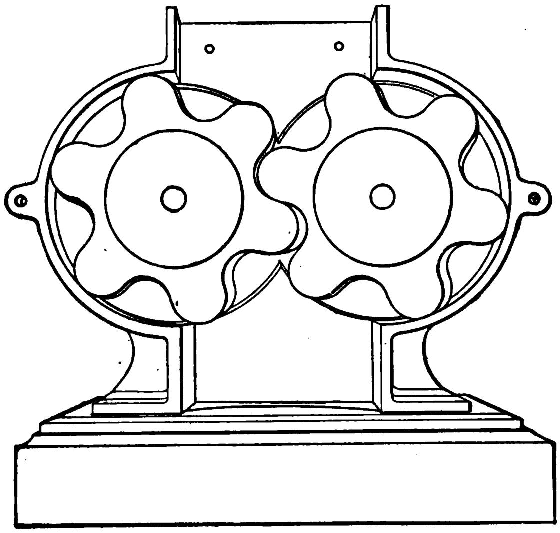

FIG. 177*a*.

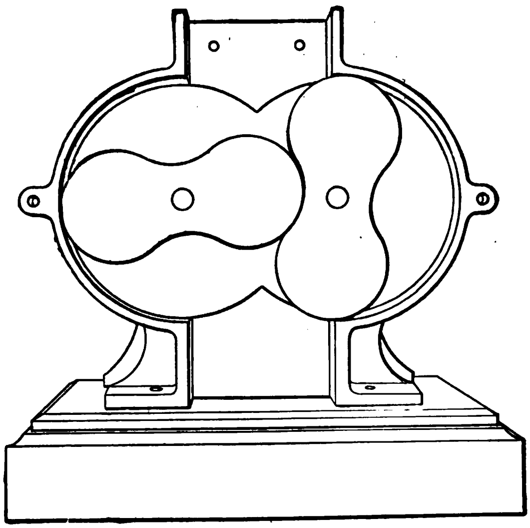

FIG. 177*b*.

Pappenheim chamber wheel-train, consisting of a pair of equal toothed wheels, having continuous tooth-contact, while the points and sides of the teeth fit as closely as possible to the walls of the chamber. This train has been utilized to a considerable extent as a pump for water, the volume discharged per revolution being evidently equal (if leakage is neglected) to twice the volume of the tooth-spaces of one wheel. Such a pump is, of course, most suitable for running at a high speed and against a low pressure. Fig. 177*b* represents the well-known Root blower, the action being identical with that of the mechanism of Fig. 177*a*. The wheels in this machine have, however, only two teeth each, and external gearing is required to maintain constant contact between the teeth. Epicyclic chamber wheel-trains are sometimes employed. Fig. 177*c* represents the mechanism of the

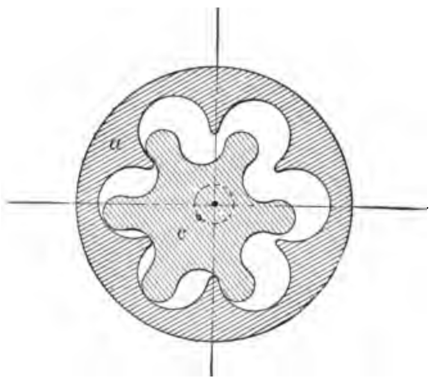

FIG. 177*c*.

Hersey water-meter. Here one wheel forms the case *a*, while the rotating shaft *b* (not shown) carries an eccentric-pin on which works the "rotary piston" or wheel *c*. It will be at once seen that this arrangement is really an epicyclic train.

Inventors have eagerly sought to discover some kind of chamber wheel-train in which the more or less imaginary disadvantages of the reciprocating engine or pump are

avoided. Almost every conceivable form of such gear* has been invented and reinvented and used with varying success. No enthusiast, however, has yet succeeded in producing a machine which is a serious competitor with the ordinary direct-acting engine or pump formed from the slider-crank chain when used for the same kind of work.

88. Ratchet-trains Containing Non-rigid Links. — The classification of Ratchet Mechanisms in general has been considered in the last chapter; we have now to study examples of those ratchet-trains which contain non-rigid links.

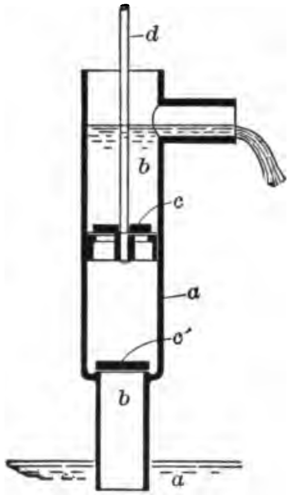

FIG. 178.

Fig. 178 shows diagrammatically the construction of a common lift-pump. On comparing Figs. 178 and 153 we find in each case a body *b* to be raised or lifted by the action of a pawl *c*, moved by a running-ratchet train. In the pump this pawl takes the form of a non-return valve carried in the pump-bucket *d*. The body of the pump corresponds to the frame *a* in Fig. 153. Further, it is plain that to keep the fluid in the pump from running back we must provide a foot-valve *c'* corresponding exactly to the pawl *c'* in Fig. 153. We have here in fact a checking-ratchet train *abc'* exactly as in the mechanism of Fig. 153. The common lift-pump is then a combination of two ratchet-trains acting on the same link, and this link is the fluid which is being pumped.

In pumps special devices are often necessary to obtain a more continuous motion of the fluid than is possible with a single-acting ratchet-train. Fig. 179 shows diagrammatically one example of this—a pump of a type sometimes used

* See Reuleaux, Kinematics of Machinery, Chapter X; Burmester, Lehrbuch der Kinematik, §§ 96–109.

for operating a hydraulic accumulator. Here the pump-piston d is provided with an enlarged rod d_1 of cross-section approximately equal to one half the area of the piston or bucket. Thus during the stroke from left to right one half of the fluid passing the valve c_2 is compelled to issue through the valve c_3, and one half enters the pump-barrel. During the reverse stroke this remaining half is expelled, and another volume of fluid enters the pump through the valve c_1. This "differential" pump, therefore, gives a fairly continuous discharge, but differs from a double-acting pump

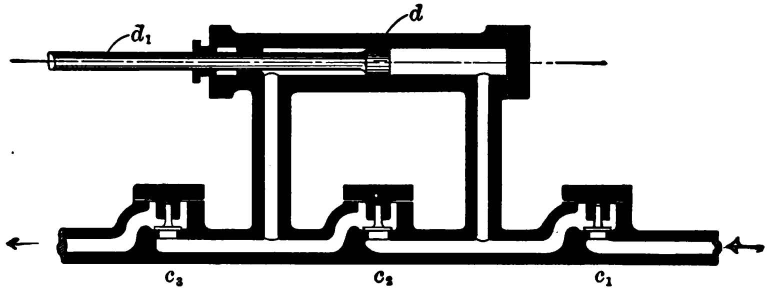

FIG. 179.

in that its suction is not continuous, but only occurs during one stroke of the bucket. We have in this case an example of the combination of three running ratchet-trains.

In Fig. 180 is shown an ingenious form of pump in which only one set of valves is required, the pump-piston itself performing the function of a releasing ratchet. The Edwards air-pump is used for pumping air and water from the condenser of a steam-engine. The bucket or plunger P has no passage through it, and during the downward stroke, while the head valves V are closed, the pressure in the space A is reduced, so that air passes in from the condenser through the ports BB as soon as these are uncovered. At the same instant the plunger on reaching the bottom of its stroke has displaced the water from the bottom of the pump and has driven it through the passage C into the space above

the bucket. The air and water are then discharged on the upward stroke. It will be seen that we have here a releas-

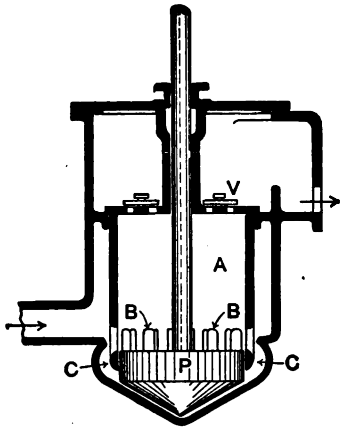

FIG. 180.

ing-ratchet train in which the ratchet (the pump-plunger) itself propels a portion of the fluid to be moved, and also prevents it from returning. When at the lowest point of its stroke the piston in uncovering the ports has acted as a

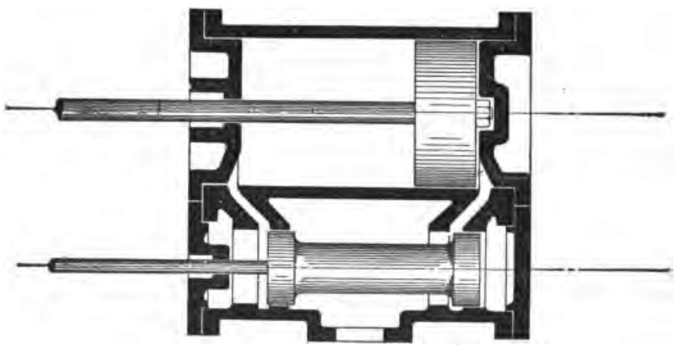

FIG. 181.

driving and releasing ratchet, first opening the passages *B* and then propelling the fluid through them.

From this point of view we may regard any sliding-valve

as a checking and releasing ratchet. Fig. 181 shows a longitudinal section through the cylinder of a steam-engine provided with a piston-valve, and it will be seen that this valve uncovers the steam-ports and admits and cuts off the steam just in the same way as the bucket of the Edwards pump uncovers its ports. The slide-valve of a steam-engine is, however, only a checking and releasing ratchet; it has no part in propelling the fluid.

Valves and cocks are frequently employed as brakes, and they then form parts of frictional ratchet-trains in which the moving link is a fluid. In Fig. 182 we have a diagram

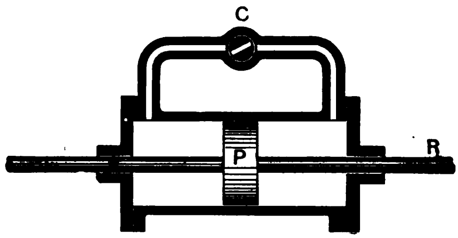

FIG. 182.

of an arrangement used to control the longitudinal movements of a rod *R*. A piston *P* works in a closed cylinder filled with fluid, and the two ends communicate by means of a passage which can be wholly or partially closed by rotating a cock *C*. A valve would, of course, answer the purpose. Here the partial closing of the cock or valve opposes a frictional resistance to the movement of the fluid, and therefore also to the movement of the piston, and in fact the cock or valve acts as a checking ratchet. Somewhat similar arrangements may be devised to act as hydraulic brakes in the case of rotating shafts; the well-known Froude brake is an example where the necessary resistance to the motion of an engine-shaft is obtained by attaching to the shaft a special rotary pump which discharges its water through a small passage.

Ratchet-trains often contain belts or other flexible links.

The strap brake of Fig. 183 may be looked upon as a frictional checking-ratchet train. The fixed link of the train is not shown, but it will be easily seen that the strap corresponds in function to the brake-block and lever of Fig. 159.

Flexible links are occasionally used in clutches, which,

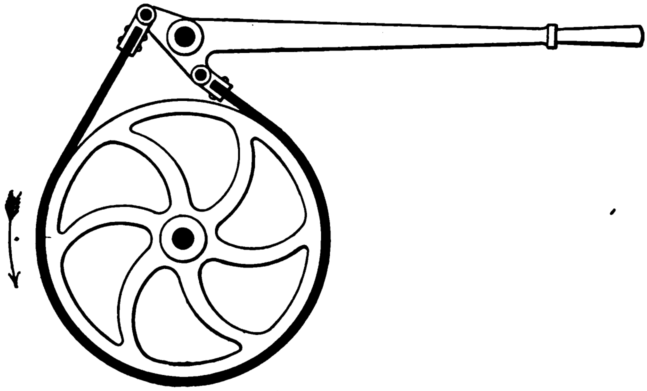

FIG. 183.

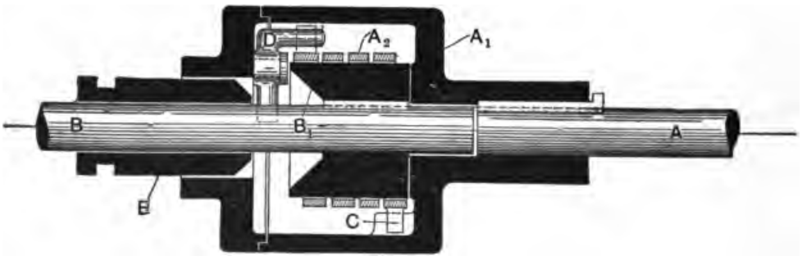

FIG. 184.

as we have already seen (§ 77), are really stationary ratchet trains. Fig. 184 shows the longitudinal section of a coil clutch, whose purpose is similar to that of the friction clutch of Fig. 159*a*.

The action of the contrivance may be explained as follows: The driving shaft A has firmly secured to it the hollow drum A_1, inside which is a metallic coil A_2 loosely enclosing the pulley B_1, which is keyed to the driven shaft B. One end of the coil (at C) is fixed to A_1, the other end has upon it

a radial projection and can be rotated so as to cause the coil to grip the outer surface of B_1. This rotation is accomplished by slightly turning the lever D on its pin by the aid of the sliding sleeve E, which can be moved along the shaft by means of a fork engaging in its groove. When pushed in, the conical end of E pushes aside the lower arm of the lever D and closes the coil. Such a clutch will only drive one way, but the numerous turns of the coil on the drum give it enormous frictional resistance, and the end pressure on the shaft A is not large. The whole arrangement forms a frictional ratchet-train.

89. Pressure Escapements Containing Fluid Links.—We have classed under the term escapements certain checking- and releasing-ratchet trains which are so arranged that the moving link is alternately released and checked by the action of the mechanism itself. Escapements containing fluid links form a class of machines which are of the utmost importance industrially, and some examples of such mechanisms will now be considered, following the nomenclature of § 78. It was there shown that escapements are really ratchet-trains which work automatically, and in the same way a self-acting fluid-ratchet-train may be said to be a fluid or pressure escapement, the driven or moving link being the working fluid.

A steam-engine or fluid motor which is provided with a governor regulating and rendering uniform its rate of motion obviously answers to our definition of a *uniform escapement.* In a properly governed steam-engine or gas-engine we may compare the function of the governor with that of the pendulum or balance-wheel of a clock or chronometer, while the escapement is evidently represented by the valve-gear. The valves themselves control the range of movement of the working fluid, exactly as the ratchets in a clock escapement control the range of movement of the escape-wheel.

Periodical fluid escapements are not of frequent occurrence. We may perhaps class under this head such contriv-

ances as gas- and water-meters, which vary their rate of motion in proportion to the quantity of fluid passing per unit of time.

Adjustable or variable fluid escapements are of considerable importance. A large number of pressure mechanisms corresponding somewhat in their mode of action to the hoisting machine described in § 78 are used as steering- or reversing-gears. Such a contrivance consists essentially of a controlled motor (*moteur asservi*) so arranged that when started by the admission of the working fluid the motor itself closes the admission-valve, and therefore stops unless the controlling valve is still further opened by hand. This

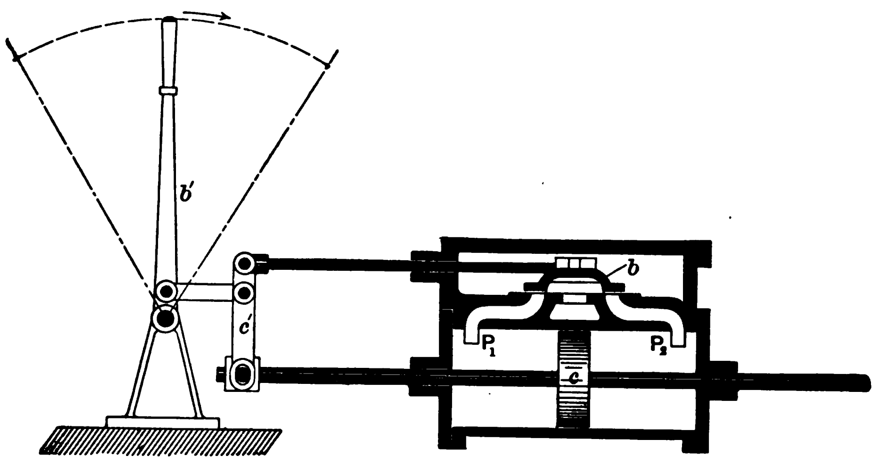

FIG. 185.

will perhaps be made clear by an example. The slide-valve controlling the admission and exhaust of steam to the cylinder in Fig. 185 is connected to a system of levers as shown. When the lever b' is moved by hand in the sense shown by the arrow, steam is admitted through the port P_1 while the port P_2 is placed in communication with the exhaust. The piston c moves in response, and, if the lever c' is properly proportioned, gives the valve b a backward motion exactly equal to the forward movement it received from the hand

lever *b′*. Thus the piston *c* follows the motion of the hand-lever *b′*. It will be obvious that kinematically this mechanism is of the same general class as the hoist previously described, and is accordingly an adjustable escapement.*

A large number of fluid-ratchet-trains and escapements are discussed by Reuleaux.†

* For a description of Brown's interesting and ingenious steering-gear, in which the whole engine is made to move and then stops itself after turning the rudder through the required angle, see *Engineering*, Vol. XLIX, p. 491.

† Constructor, §§ 319–332.

CHAPTER XI.

CHAINS INVOLVING SCREW MOTION.

90. Formation of Screw Surfaces.—It has already been stated (§ 8) that lower pairs of elements can be constructed in which the surfaces in contact are screws of uniform pitch. Fig. 186 serves to illustrate the formation of

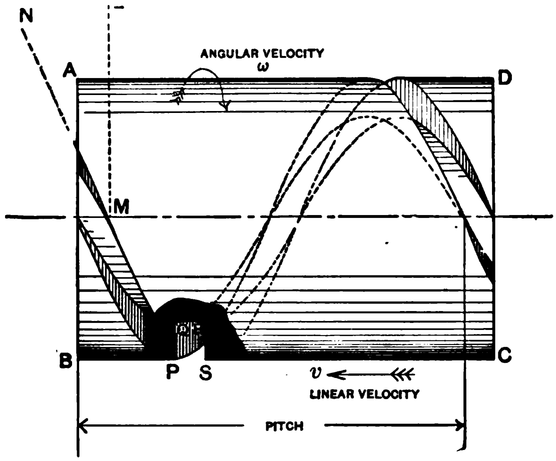

FIG. 186.

such surfaces. Imagine that a cylinder $ABCD$ is caused to rotate with uniform angular velocity, as indicated by the arrow, and let a cutting tool whose point is ground into the shape $PQRS$ be moved with uniform linear velocity v in a direction parallel to the axis of the cylinder, so as to cut out a continuous groove in the material of the cylinder. If now the tool is so set that the lines PQ and SR when pro-

duced pass through the axis of the cylinder, the surfaces forming the side of the groove will be screw or helical surfaces of uniform pitch. It will evidently be possible to form in a somewhat similar manner a hollow cylinder having the material of its inner surface removed in such a way as to leave a projecting thread of such a form as will exactly fit into the groove $PQRS$. The inner surface of this nut will be the exact counterpart of the outer surface of the screw, and when working together their relative motion must be a copy of the original relative motion of the cutting tool and the cylinder. In other words, the only possible relative motion of such a screw and its nut will be a motion of rotation, combined in a constant ratio with a motion of translation along the axis of rotation. By the term *pitch* we mean the distance (measured along the axis of rotation) through which the nut moves relatively to the screw during one complete relative rotation. Thus if ω be the angular velocity of the cylinder in radians per second, the time of one complete rotation will be $\frac{2\pi}{\omega}$ seconds. During this time the cutting tool will have moved a distance $\frac{2\pi v}{\omega}$; this expression therefore gives the numerical value of the pitch. If we imagine that a piece of paper wrapped round the cylinder has the outline of the screw-thread marked upon it, and is then unwrapped, the line representing the edge of the screw-thread will be found to be straight, and it will make with the line representing the edge AB of the cylinder an angle such as LMN. A little consideration will show that the tangent of this pitch-angle will be

$$\frac{\text{pitch of thread}}{\text{circumference of cylinder}}.$$

It is quite easy to arrange a mechanism which will cut a screw-thread of *variable pitch*. This is, in fact, often done in rifling guns. In this case, if the angular velocity of the screw is uniform, the linear velocity of the tool must be

variable, and the pitch-angle changes as we go along the thread. A hollow surface the exact counterpart of the screw would then only fit exactly in one position, and no relative motion of such a pair of surfaces would be possible. It is for this reason that a screw pair composed of rigid elements must consist of screw surfaces of *uniform pitch*. The section of the thread, as governed by the form of the cutting tool producing it, may be of any convenient form, and a number of standard threads are described in text-books on machine design. The reader should note that screws are often made with two, three, or a larger number of threads by cutting the required number of independent grooves on the cylinder. These threads may further be either right- or left-handed. The thread in Fig. 186 is right-handed; Fig. 187 shows a left-handed screw having

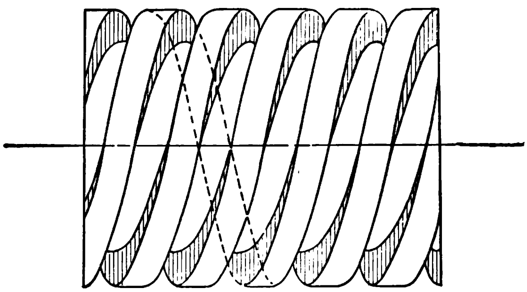

FIG. 187.

three threads. We shall see later that these multiple-threaded screws are of importance in screw mechanisms involving higher pairing, and we now consider certain cases in which lower pairing of screw surfaces is used in chains containing rigid links only.

91. Screw Mechanisms Involving Lower Pairing of Rigid Links.—The relative motion of screw links is in general non-plane. On examination it will be found that in a screw and its nut, while there is at any instant rotation about the axis of the screw, there is also a simultaneous linear movement along that line. In more complex cases of the screw

motion of two bodies it has been pointed out * that there is at any instant a line common to the two bodies, called the *twist axis*, about and upon which each body is (at the instant

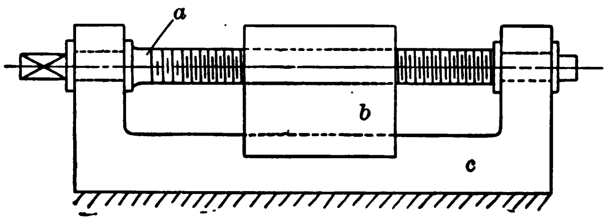

FIG. 188.

considered) turning and sliding relatively to the other body. In this work but little space can be devoted to the consideration of relative motion of this complex character, and in this section we shall discuss some of the simpler screw mechanisms involving lower pairing.

The simplest screw chain is shown in Fig. 188; it comprises three pairs—a screw pair *ab*, a turning pair *ac*, and a sliding pair *bc*. This chain is of common occurrence in the form of a screw press. By a suitable choice of the pitch of the screw we can obtain a machine in which a large angular motion of *a* gives us a comparatively small linear motion of *b*, so that in a copying-press, for instance, a large pressure is obtained by applying a relatively small force to the end of the screw arm.

If the screw has a sufficiently fine pitch this machine cannot be reversed; that is, it is not possible by the application of an axial force to the nut *b* to cause rotation of *a*. By making the pitch of the screw sufficiently great, however, this action becomes possible, as in the common Archimedean drill.

A little consideration will show the reader that we may look upon a sliding pair as a screw pair of infinite pitch, while a turning pair is also a special case of a screw pair in which the pitch is zero. Accordingly we may expect to

* Kennedy, Mechanics of Machinery, § 68.

find mechanisms of three links containing two screw pairs and a sliding pair, or two screw pairs and a turning pair (as shown in Figs. 189*a* and 189*b*), the pair *bc* or *ac* in Fig.

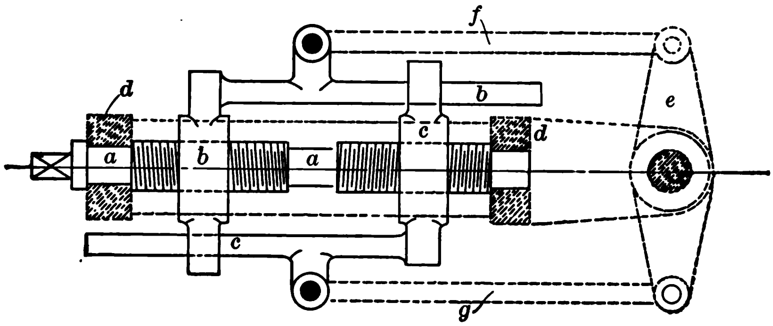

FIG 189*a*.

188 having been modified into a screw pair. Further, it is possible to transform the last remaining turning pair of Fig. 189*b* into a screw pair and obtain a chain of three links and three screw pairs. The reader should have no difficulty in sketching for himself such a chain.

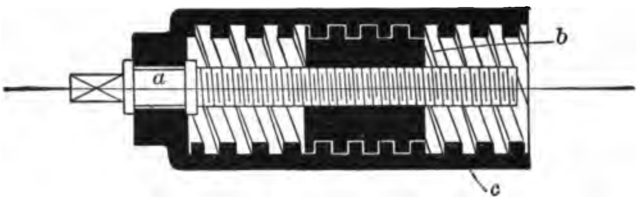

FIG. 189*b*.

Fig. 189*a* represents the chain containing two screw pairs and a sliding pair, as employed in a form of steering-gear; the complete gear forms a compound chain of seven links, four of which (shown by dotted lines) are added to the screw chain itself. The screw *a*, on which are cut two separate threads, right- and left-handed respectively, gears with two nuts *b* and *c* which evidently have a relative sliding motion, approaching or receding from each other as *a* rotates. The links *f* and *g* connect *b* and *c* to the arms of a

yoke *e* secured to the rudder head. The frame or fixed link *d* is the hull of the ship, to which are fixed the bearings in which *a* and *e* rotate. Plainly, rotation of *a* will cause the rudder to turn.

Fig. 189*b* shows the chain containing two screw pairs and a turning pair. Its most important application in practice will be discussed when we deal with screw chains involving fluid links. (See § 92.)

A great variety of more complex screw chains are in practical use. Fig. 190 shows a *crossed screw chain* often employed as a portion of the reversing-gear of steam-engines. It consists of five links and contains a screw pair *ab* and

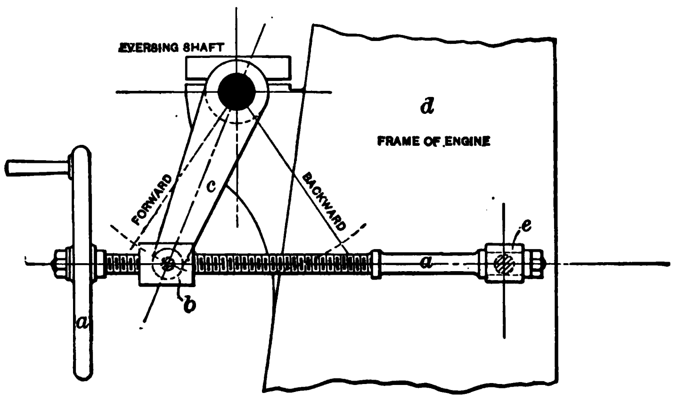

FIG. 190.

four turning pairs. The rotation of a hand-wheel on *a* moves the reversing-shaft from its position when the engine goes ahead to the position when the engine is in backward gear.

The mechanism of Fig. 190 is a simple example of a class of mechanisms involving *general screw motion*, in which the relative motions of the links are often very complex. The majority of such chains, in fact, have not been worked out kinematically, but the more complicated general screw mechanisms find so small a field of usefulness that we shall not devote any space to them here.

92. Screw Mechanisms containing Fluid Links. — One of the simplest screw mechanisms containing a fluid link is the rifled gun shown in longitudinal section in Fig. 191. This train consists essentially of three links. We have the gun itself, a, having traced upon the surface of its cylindrical bore the rifling, in the shape of a many-threaded hollow screw shown in cross-section at AB. The projectile or shell b is introduced at the breech of the gun, which is closed by a screw-plug or breech-block, and the projectile is provided at its base with one or more copper driving-bands, which are

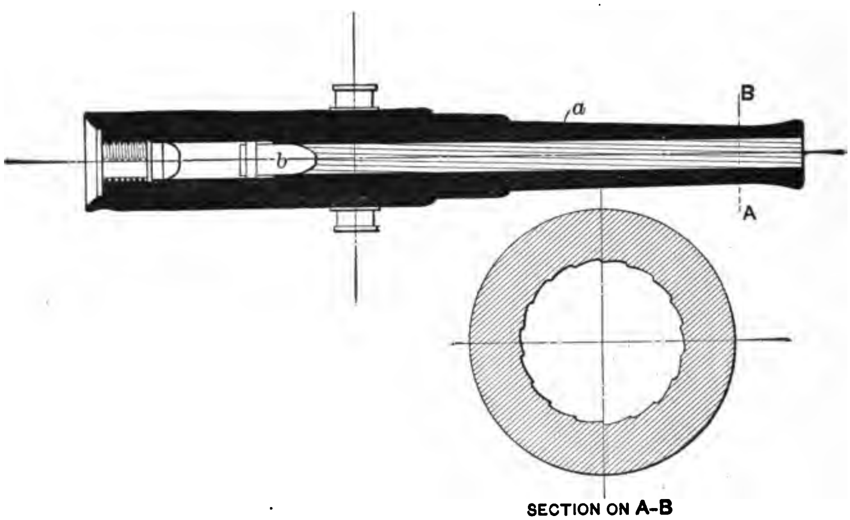

FIG. 191.

of such a size that when the projectile is forced through the bore the projecting portions of the rifling cut into the copper, and in this way cause the shell to rotate. The third link, c, is of course the gas which is produced by the combustion of the charge, and which exerts the pressure necessary to propel the shell. It should be noted that the pitch of the rifling has to be large, compared with the calibre or diameter of the bore of the gun. Frequently the pitch of the rifling is not uniform, but is so designed as to decrease from the breech

to the muzzle in such a way as to give as nearly as possible uniform angular acceleration to the shell.

In Fig. 192 is represented a mechanism which is a special case of the screw-chain used for such important purposes as the propulsion of ships (screw propeller), the measurement of speed through fluid (anemometer, patent log), the

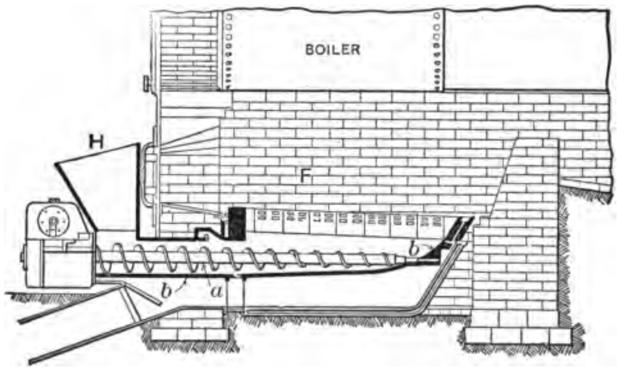

FIG. 192.

utilization of the energy of the wind (windmill), and so on. The figure shows diagrammatically a screw conveyor used for the purpose of forcing broken coal from the hopper *H* into the furnace *F* of a mechanical stoker. We have here a rotating propeller *a* of peculiar form; it is contained in a casing *b*, and acts on the powdery material enclosed by the casing. The reader will note that in the screw propeller of a ship we have exactly the same mechanism, except that the outer casing is used only under special circumstances,* and the material acted upon is fluid.

The mechanism of Fig. 189*b*, when modified by the substitution of a fluid link for the piece *b*, takes the form of the parallel-flow (Jonval) turbine of Fig. 193, and is used for purposes of motive power. Here *c*, the turbine-casing, carries a bearing for *a*, the hollow shaft, and also has upon it a number of fixed guide-blades corresponding kinematically to the hollow screw-thread of Fig. 189*b*. The fluid,

*Barnaby, Marine Propellers, Chapter VII.

rushing past these blades, encounters the blades of the turbine-wheel *a*, to which it communicates motion. The kinematic correspondence of the two mechanisms is evident.

It should be noted that the surfaces of the guide-blades and buckets of a turbine, or of the blades of a propeller,

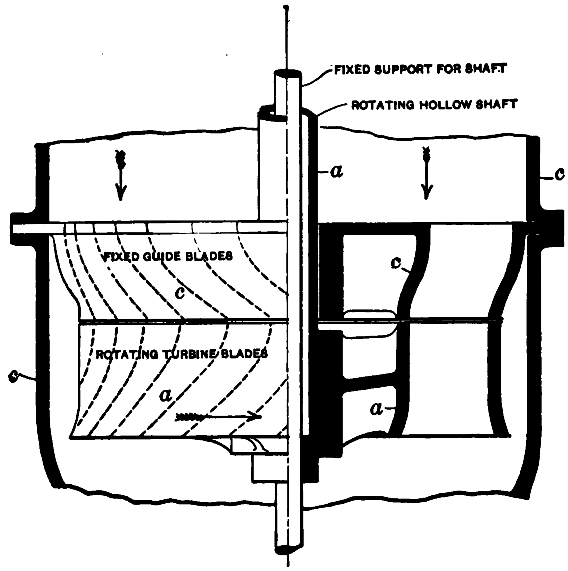

FIG. 193.

are not necessarily true helical surfaces. With solid links we have seen that in order to obtain lower pairing the screw surfaces must have uniform pitch. The adoption of a varying pitch in the rifled gun is only possible because the copper driving-band, which pairs with the rifling, is narrow and so soft as to be deformed with comparative ease. When we consider the pairing of fluid links with such surfaces, however, the mobility of the fluid permits of great latitude in the form of the curved surface over which it flows.

We have so far considered only screw-threads traced upon a cylinder, but there is no reason why such threads should not be formed on a conical surface, or indeed upon many surfaces of revolution. Fig. 194 shows a thread cut

upon a globoid, for instance, Let us now imagine a screw-thread traced upon a conical surface, as is the case in some forms of self-centring chuck,* or in the breech-blocks of certain quick-firing guns.† From such a screw-thread it is

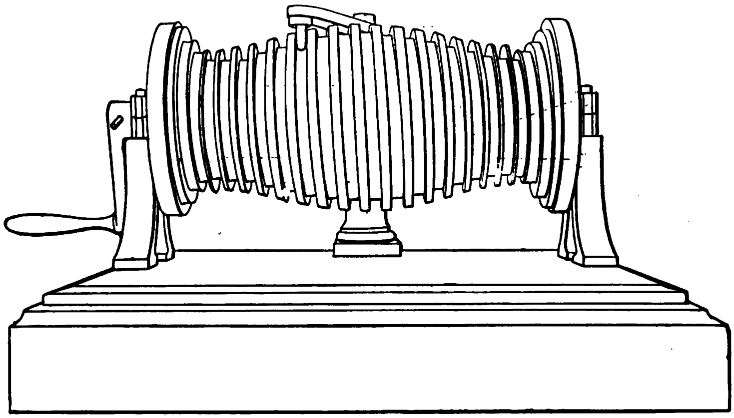

FIG. 194.

but a step to the formation of such a surface as that of the vane of the wheel of a centrifugal pump (Fig. 176) or the vane of a radial-flow turbine, where the blades form what may be termed a screw surface projected on a plane. The kinematic chain of Fig. 176 is then really a modification of that shown in Fig. 193, the guide-blades being suppressed, and the whole forming a pump instead of a motor. The curves of the blades in a centrifugal pump are formed in such a fashion that their rotation impels the fluid from the centre to the outside of the pump-casing. They are thus spiral in form, or may even take the shape of radial straight lines.

93. Screw-wheels and Worm-gearing.—In machine construction screws are employed not only in lower pairing for driving, or being driven by, rigid nuts, but also, in higher pairing, for gearing with rotating toothed wheels. In this case contact between the screw and the link with which it

* "Horton" chuck. † *Engineering*, Vol. LXI. p. 11.

pairs takes place either along a line or at a point. The ordinary worm and worm-wheel is the most familiar example of such gearing. Fig. 195 represents in plan and elevation

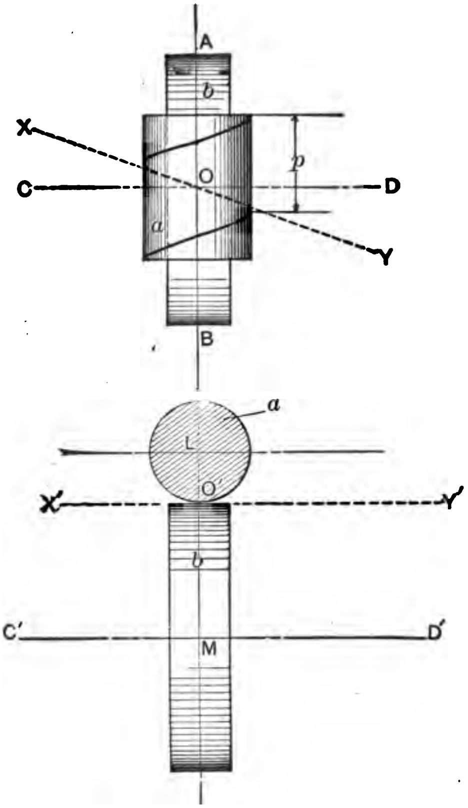

FIG. 195.

two cylindrical wheels, a and b, whose axes AB and CD do not intersect and are at right angles in plan. The wheels are in contact at the point O through which passes $LO'M$, the common perpendicular to AB and CD. The length of this common perpendicular is of course the sum of the radii of the two cylinders. Let a helical line or screw-thread be traced on the surface of a, so as to pass through the common point O, the pitch-angle of this helix being XOC. Also

suppose that a second helical line, not shown on the diagram, of pitch-angle XOA, is traced on the surface of b, so as also to pass through the point O. The two helices will then touch at that point, and the line XOY will be their common tangent. If now the helix on b is replaced by a projecting thread, while that on a is converted into a corresponding groove into which the thread gears, any rotation of a about its axis AB will cause the rotation of b about its axis CD, and this relative motion of a and b will be continuous if we provide a series of projecting threads on b so spaced as to come into gear in succession with the thread or groove on a.

It will be noted that, in Fig. 195, a is a single-thread screw, while the wheel b is a portion of a many-threaded screw, the number of threads on b being equal to the number of times that the pitch p is contained in the circumference of b. We can, however, evidently make pairs of screw-wheels in which a as well as b is a portion of a many-threaded screw, and a pair of such wheels is shown in Fig. 196, the teeth or threads being represented by the inclined lines. In speaking of the pitch of the teeth of these wheels, we must distinguish between (1) the helical pitch, or pitch of the screw-thread (p in Fig. 195); (2) the normal pitch, or distance from centre to centre of teeth, measured at right angles to their length (q in Fig. 196); (3) the circumferential pitch (r, Fig. 196); (4) the axial pitch, or distance from centre to centre of teeth measured parallel to the axis of the wheel (s, Fig. 196). A little consideration will show that in a pair of screw-wheels the circumferential pitch of each must be equal to the axial pitch of the other, supposing that, as in the figure, the axes of the wheels are at right angles in plan.

We have now to find the angular velocity ratio of the wheels a and b. It is plain that since the teeth of a and b, while the wheels rotate, remain in continuous contact, their velocity measured along a line drawn perpendicular to their common tangent at the point of contact and lying in the plane which touches both wheels must be equal. In Fig. 197

this common velocity is represented by the line v_c. Now let ω_a and ω_b be the angular velocities of a and b respectively, while r_a and r_b are their radii. Then if v_a, v_b are the actual linear velocities of points on the pitch-circles of a and b, we have

$$v_a = \omega_a r_a \quad \text{and} \quad v_b = \omega_b r_b.$$

The lines OL and OM in the figure are supposed to be

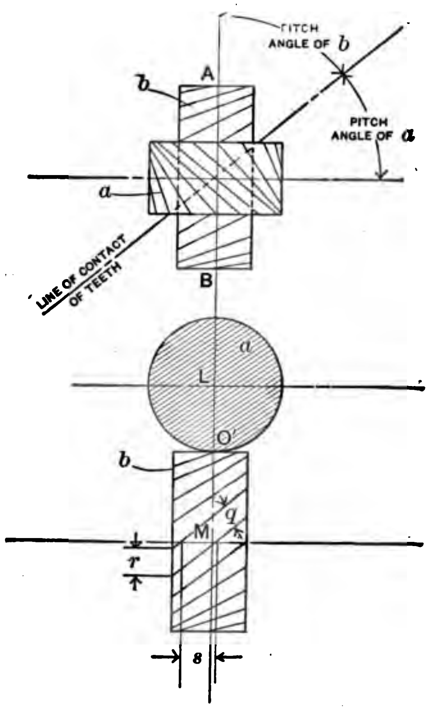

FIG. 196.

drawn in the plane touching both wheels, and represent in magnitude and direction the linear velocities of the respective pitch surfaces. ON is drawn in the same plane, and represents v_c, the common velocity of the teeth of both wheels measured in a direction perpendicular to the common tangent of the teeth.

The velocity v_a may be resolved into two components.

namely, ON, the velocity of a point on the tooth of a resolved in a direction normal to the line of the tooth, and NL, the velocity resolved along the line of the tooth. Similarly OM

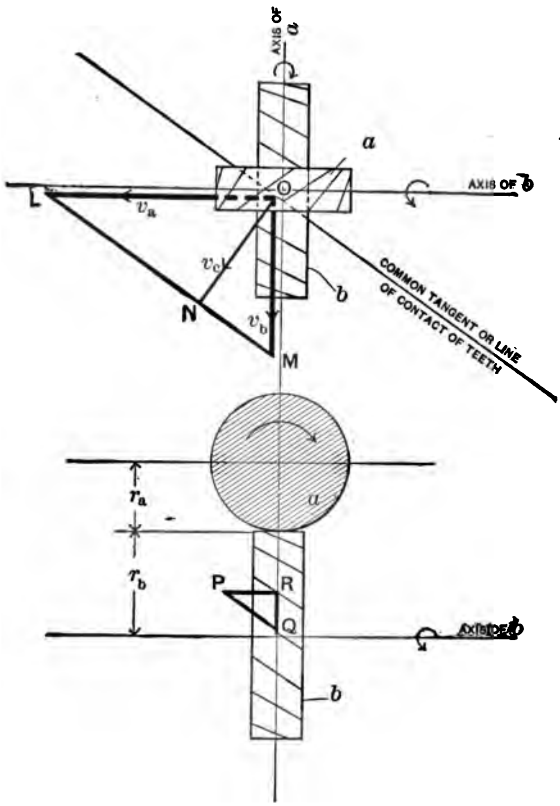

FIG. 197.

may be resolved into two components at right angles, ON and NM; as we have seen, the component normal to the line of contact of the teeth must be the same for both wheels, because it represents the common velocity v_c. The line LM will represent the relative linear velocity (along the tooth) of the pitch surface of the wheel a relatively to that of the wheel b, or the speed with which the teeth slide lengthways over each other.

Now consider the circumferential and axial pitches of the two wheels. From the figure, by similar triangles

$$\frac{v_a}{v_b}=\frac{OL}{OM}=\frac{PR}{QR}=\frac{\text{axial pitch of } b}{\text{circumferential pitch of } b}$$
$$=\frac{\text{circumferential pitch of } a}{\text{circumferential pitch of } b},$$

and

$$\frac{\omega_a}{\omega_b}=\frac{v_a}{v_b}\cdot\frac{r_b}{r_a}=\frac{\text{circumf. pitch of } a}{\pi r_a}\times\frac{\pi r_b}{\text{circumf. pitch of } b}$$
$$=\frac{\text{number of threads on } b}{\text{number of threads on } a}.$$

It is thus seen that in screw gearing of this kind the velocity ratio is independent of the sizes of the wheels, and depends solely on the number of threads with which they are provided.

A particular form of screw-gearing is frequently employed to transmit motion with a high velocity ratio between shafts at right angles in plan. The smaller wheel has only one, two, or three threads, of small axial but large circumferential pitch, and is known as a *worm*, while the *worm-wheel* has many teeth, of small circumferential but large axial pitch. The velocity ratio is, as we have just seen, simply the inverse ratio of the number of threads. Worm-wheels of good design have the form of their pitch surfaces modified so that their teeth are no longer screw-threads traced on a cylindrical surface, but are formed so as to obtain a larger area of contact between the teeth than would be possible in the case of a cylindrical screw-wheel.* The teeth of a pair of accurately formed cylindrical screw-wheels of rigid material would only touch in a point; in practice there would of course be a very small but perceptible area of contact. Such wheels are therefore most suitable for light loads; and for heavy service, worm-gearing, in which the screw-thread and wheel-tooth may have line

* See § 94.

contact, is preferable. Fig. 198 shows the appearance of screw-gearing and worm-gearing as actually made.

The axes of a pair of screw-wheels may of course make any desired angle in plan with one another. In a given case, when this angle, the sum of the radii of the pitch surfaces, and the velocity ratio have been decided, a number of different pairs of wheels may be designed which will obtain

FIG. 198.

the intended result; the difference between them depending on the pitch-angles which are selected. A numerical example will make this clearer. Suppose that screw-gearing is to be designed to connect two shafts making an angle of 60° in plan, the shortest distance between the axes being 10 inches, and the velocity ratio three to one. Two solutions of this problem are shown in Fig. 199. In the first case a pitch-angle of 60° has been chosen for each wheel; in the second case the line of contact of the wheel-teeth is parallel

to the axis of b, and the pitch-angles of the wheels a and b are respectively 30° and 90°. The wheel b is thus a spur-wheel. The velocity diagrams showing the relation between v_a, v_b, and v_c are drawn in the two cases. In order to determine the radii of the pitch surfaces of the wheels, we have in the first case

$$v_a = v_b$$

But $\frac{r_a}{r_b} = \frac{v_a}{v_b} \cdot \frac{\omega_b}{\omega_a}$ and $\frac{\omega_b}{\omega_a} = \frac{1}{3}$. Hence

$$\frac{r_a}{r_b} = \frac{1}{3}.$$

The distance between the axes being 10 inches, plainly $r_a = 2.5''$ and $r_b = 7.5''$.

In the second case $v_a = 2v_b$; hence, since $\frac{\omega_b}{\omega_a} = \frac{1}{3}$ as before,

$$\frac{r_a}{r_b} = 2 \times \frac{1}{3} = \frac{2}{3},$$

so that $r_a = 4''$ and $r_b = 6''$.

In both cases the number of teeth on the wheels a and b must be in the ratio 1 : 3, but in the first case the wheels have the same circumferential pitch, and the radii are therefore inversely as the angular velocities; while in the second case the circumferential pitches of the wheels a and b are in the ratio 2 : 1, so that the radii are in the proportion 2 : 3. From the diagrams it is evident that the sliding velocity of the teeth will be in the first case

$$v_s = v_a = v_b,$$

and in the second case

$$v_s = v_a \times \frac{\sqrt{3}}{2} = v_b \times \sqrt{3}.$$

It is noteworthy that for the same speeds of the shafts in the two cases v_a will be greater in the second instance than in the first in the proportion of 4 : 2.5. Hence the sliding velocity of the teeth will be greater in the second design in the proportion $2\sqrt{3}$: 2.5, or nearly 1.39 : 1. This is shown in the

figure, where the velocity diagrams, shown in heavy lines, are both drawn to the same scale. In screw-wheels the sliding velocity of the teeth is a minimum when the wheels have the same pitch angle.

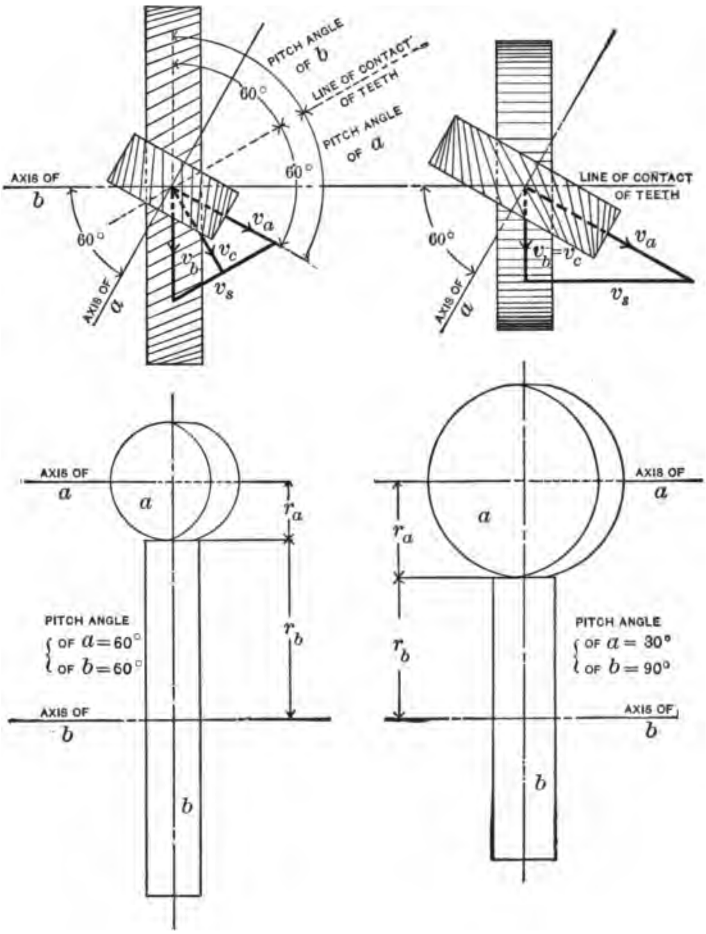

FIG. 199.

94. Forms of Teeth in Screw-gearing and Worm-gearing.—The cylinders shown in the previous figures are of course intended to represent the pitch surfaces of the actual screw-

wheels. We have seen that the relative linear motion of these surfaces in the plane which touches them both is composed of a relative sliding motion along the line which is the common tangent to the teeth at the point of contact, and a common movement along a line perpendicular to that common tangent. If we consider only this common velocity, it becomes plain that we have here a case similar in some respects to that of a pair of spur-wheels. In spur-gearing the teeth have a common velocity along a line which is perpendicular to the line of centres and to the line of the teeth, and the relative motion is simply a rolling together, the virtual axis being a line parallel to the wheel axes and passing through the pitch-point. In a pair of screw-wheels the relative motion is also one of rolling; the virtual axis being the common tangent (XOY, Fig. 195), but this motion is combined with a relative sliding along the common tangent. This line may therefore be considered as the *twist axis** of the two wheels, and it may be shown that the *twist axodes* of a pair of screw-wheels form a pair of hyperboloids;† the two surfaces rolling together and at the same time sliding along their line of contact. If the axes of the screw-wheels are parallel, the hyperboloids become cylinders and there is no sliding motion along the line of contact.

The above principles must be considered in determining the proper profiles for the teeth in screw-wheels. We must in fact imagine that the pair of pitch surfaces are cut by a plane passing through the point of contact and perpendicular to the common tangent. We must then take such shapes for the teeth as would work together correctly if formed on a pair of pitch-circles having the same radii of curvature as the sections—in which the actual pitch surfaces are cut by our imaginary plane—have at their point of contact.

Evidently the traces of the cylindrical pitch surfaces of the screw-wheels on the imaginary plane will be a pair

* See § 91. † See § 95.

of ellipses, always touching at the ends of their minor axes. The plane will make with the median plane of each wheel an angle $(90° - \alpha)$, where α is the pitch-angle of the helical teeth or screw-threads, and the semi-minor axis of the ellipse will be r; the semi-major axis being $\frac{r}{\sin \alpha}$, where r is the radius of the pitch surface of the wheel. In Fig. 200 is shown the pitch surface of a screw-wheel (radius r) having traced upon its surface a helix of pitch-angle XOC. As in Fig. 195, the line XOY is a tangent to the helix at the point O, and if O were the point of contact of the wheel with another, XOY would represent the common tangent or line of contact of the teeth.

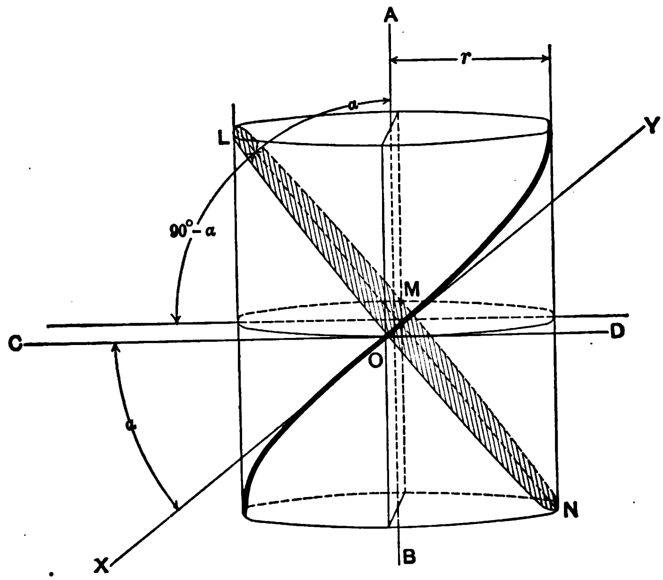

FIG. 200.

A plane passing through O and perpendicular to XOY will cut the cylinder in the ellipse $LMNO$, of which the major and minor axes will evidently be $\frac{2r}{\sin \alpha}$ and $2r$ respectively. In order to find the proper form of tooth profile (taken of

course in the plane of $LMNO$ and at right angles to the centre line of the tooth), we must take as an imaginary pitch-circle the circle of curvature of the ellipse at the point O. The radius of curvature of the ellipse at this point is easily shown to be $\frac{r}{\sin^2 \alpha}$; hence in a pair of screw-wheels the teeth profiles should be designed as if they belonged to a pair of ordinary spur-wheels of radii

$$\frac{r_a}{\sin^2 \alpha_1} \quad \text{and} \quad \frac{r_b}{\sin^2 \alpha_2},$$

where α_1 and α_2 are the pitch-angles of the teeth of the screw-wheels a and b.

Screw-wheels are often used to connect shafts which are parallel. In this case the pitch-angles of the wheels are of course equal, and if the wheels are properly designed there is no difficulty in having two or more pairs of teeth in contact at once. In order to avoid the prejudicial effect of the end thrust developed when a pair of such wheels are doing heavy work, it is usual to make each wheel of two parts, similar in pitch, but one half right-handed and the other half left-handed. Fig. 201 shows such a double helical pinion of the form employed in a rolling-mill. The teeth profiles in such wheels, taken on a section by a plane perpendicular to the axes, will be the same as those required for spur-wheels of the same diameter and circumferential pitch of teeth; for in these wheels there is no sliding motion along the teeth. If properly shaped, the teeth will be in contact along short lines inclined more or less towards the pitch surfaces, according to the pitch-angle chosen for the helices. When this pitch-angle is 90° the wheels become spur-wheels, in which the lines of contact of the teeth are of course parallel to the pitch surfaces.

When worm-gearing is constructed with cylindrical pitch surfaces, and teeth of uniform cross-section, contact, as in the case of other screw-wheels, occurs at a point only, and the forms of teeth may be found by the method just

described. It is not difficult, however, by modifying the form of the wheel, to obtain worm-gearing having linear contact. The method of doing this is fully explained in works on Machine Design.*

The section of the worm and wheel by a plane perpendicular to the axis of the wheel, and passing through the axis of the worm, is that of a rack in gear with a spur-wheel, and the form of the worm-thread and wheel-teeth in this plane may be drawn by the methods already discussed in §§ 66 and 67. The trace of such a plane is shown in Fig. 202

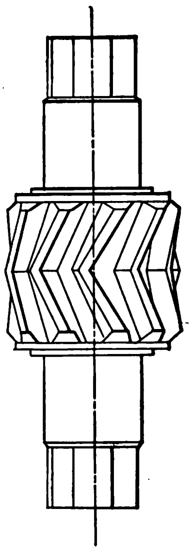

FIG. 201.

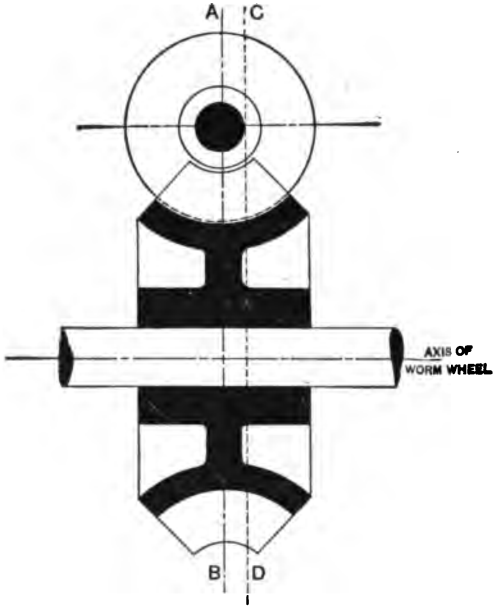

FIG. 202.

by the line AB; the figure represents the section of the worm and wheel by a plane containing the axis of the wheel and perpendicular to the axis of the worm. The form of a section of the worm by a plane parallel to the axis of the worm, and perpendicular to the axis of the wheel, is next to be determined; the trace of such a plane on the plane of the

* Unwin, "Machine Design," Vol. I, § 234.

figure is the line CD, and its intersection with the worm-thread will take the shape of a rack having curved and unsymmetrical teeth. The form of worm-wheel tooth required to gear correctly with such teeth must then be found by the proper method of construction, and the shape determined is to be used for the section of the worm-wheel tooth cut by the plane CD. A number of such sections, found for planes at different distances from the median plane AB, will enable a practically correct wheel-pattern to be made. As a rule such wheels are machine-cut by being rotated in correct relation to a steel cutter or hob which is a duplicate of the worm to be used.

The Hindley worm * has a screw-thread of varying section traced on a non-cylindrical pitch surface whose outline is an arc of the pitch-circle of the wheel. This form of tooth,

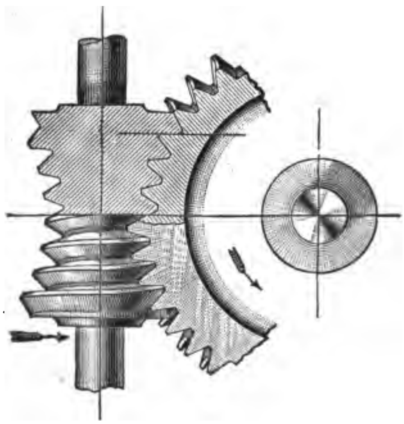

FIG. 203.

if correctly cut by means of a hob, has line contact: the teeth touching the threads at all points in the median plane.

95. **Hyperboloidal Wheels.** — It is possible to construct wheels which will transmit motion between inclined non-intersecting axes, and which are so formed that their teeth are straight and have line contact. The pitch surfaces of

* *American Machinist*, March 25, 1897.

such wheels are hyperboloids of revolution, as has already been stated. In Fig. 204 let AB and CD be the axes of a pair of such wheels; the line of contact of their teeth is to be the line XOY, passing through a point O on LOM the common perpendicular to AB and CD. In general XOY will be parallel neither to AB nor to CD.

If now we imagine the line XOY to be rotated around AB as an axis, while its position in relation to AB remains unaltered, XY will describe in space the hyperboloid a; and similarly, if we suppose the rotation to take place about CD, the hyperboloid b will be described. The two hyperboloids will of course touch along the line XY, which is in fact their twist axis when relative motion occurs. The smallest circular sections of the hyperboloids are known as the gorge-circles. We proceed to determine the relative angular velocity of a pair of such hyperboloidal surfaces, supposing that they roll together. It is to be particularly noted that hyperboloidal wheels differ from the cylindrical screw-wheels hitherto discussed, in that the pitch surfaces of the latter can touch only at a point, while those of the former are in contact along a line. The relative motions of the two kinds of wheels are, however, of the same kind, namely, a rolling together, combined with relative sliding along the line of the teeth. Let θ_1, θ_2 be the angles (in plan) made by the projection of the line of contact XY with the projection of the axis of a and the projection of the axis of b respectively. The angular velocity-ratio of the wheels a and b must evidently be the same as that for a pair of screw-wheels of the same size as the gorge-circles of the hyperboloids and having the same obliquity of teeth. The velocity diagram will therefore be that drawn in thick lines, by the method of § 93, and we shall have

$$\frac{\omega_a}{\omega_b} = \frac{v_a}{r_a} \cdot \frac{r_b}{v_b} = \frac{r_b \cdot v_c}{\cos \theta_1} \cdot \frac{\cos \theta_2}{r_a v_c} = \frac{r_b \cos \theta_2}{r_a \cos \theta_1}.$$

Now consider any point Y on the line of contact XOY. The normal to the curved surface of the hyperboloid at any

point must pass through the axis; hence a straight line drawn through Y and normal to the curved surfaces which

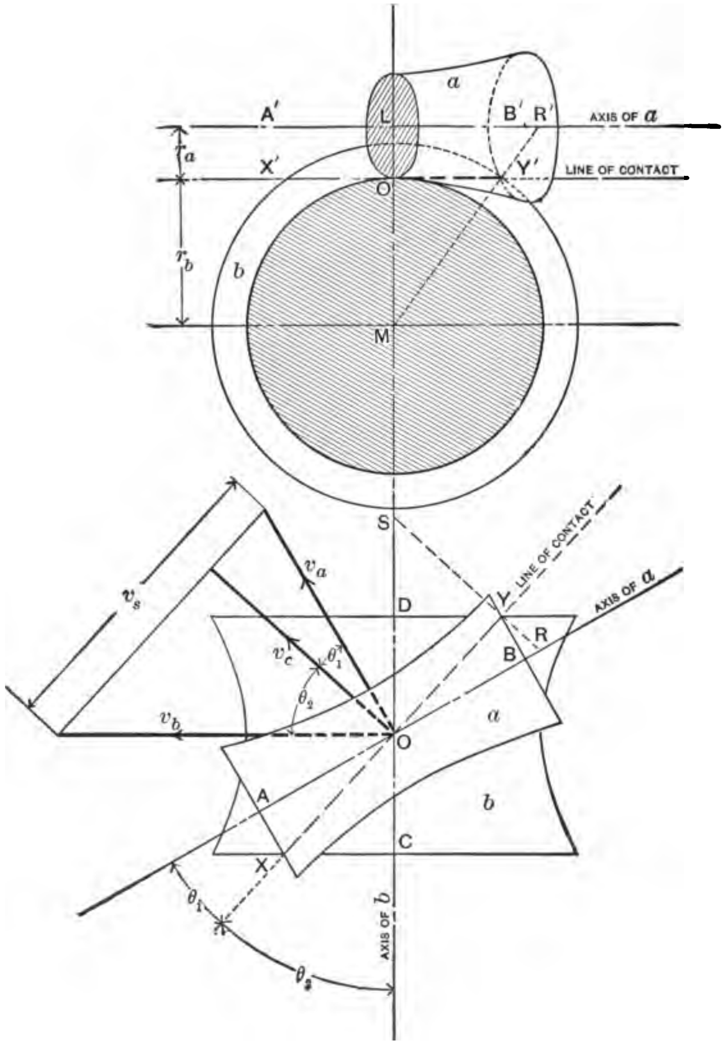

FIG. 204.

touch there must intersect both CD and AB. Such a common normal is shown in the figure, where RYS is its pro-

jection on a plane parallel to both axes AB and CD, and $R'Y'M$ is its projection on a plane perpendicular to the axis of b. In this view of course the points D, M, C, and S coincide.

By a well-known principle of projection the real lengths of the segments of the common normal are proportional to their projections RY, YS, and $R'Y'$, $Y'M$. Hence

$$\frac{r_a}{r_b}=\frac{LO}{OM}=\frac{R'Y'}{Y'M}=\frac{RY}{YS}$$

and
$$\frac{\omega_a}{\omega_b}=\frac{YS}{RY}\cdot\frac{\cos\theta_2}{\cos\theta_1}.$$

Again, since the common normal is perpendicular to XOY, the line of contact, its projection RYS will be perpendicular to the projection of XOY on a parallel plane; so that in the lower figure OYS and OYR are right angles. Thus, finally,

$$\frac{\omega_a}{\omega_b}=\frac{DY}{BY}.$$

This shows that the angular velocities of a pair of hyperboloidal wheels are to each other in the inverse ratio of the lengths of the projections of the perpendiculars drawn from any point on the line of contact to the axes; these projections being upon a plane parallel to both axes and to the line of contact.

It should be noted, that in designing hyperboloidal wheels, if the angle between the axes (in plan) and the velocity ratio are given, the position of the line of contact (in plan) is determined. Thus in drawing such a pair of wheels we proceed as follows:

(1) Draw the axes in elevation and in plan.

(2) The velocity ratio being given, draw the line of contact XOY (in plan), determining the point Y by marking off DY and BY having lengths in the proper ratio.

(3) Draw SYR perpendicular to XOY, and also draw $MY'R'$, the projection of SYR on a plane perpendicular to the axis of b.

(4) Through Y' draw $X'O'Y'$; this determines the values of r_a and r_b and settles the sizes of the gorge-circles.

(5) Proceed to complete the projections of the hyperboloids, as shown.

As in the case of screw-wheels, the velocity diagram shows the rate at which the teeth of a and b slide along each other. This relative sliding velocity is shown as v_s in Fig. 204.

It is not necessary in practice to use more than a comparatively small portion of the hyperboloid for a working wheel. Fig. 205 shows a pair of hyperboloidal rollers, and a pair of skew-bevel wheels having the same velocity ratio. It will be seen that these bevel wheels correspond in fact to the end portions of the hyperboloids. The forms of the teeth of hyperboloidal wheels may be conceived as being marked out upon the cones whose surfaces are normal to those of the hyperboloids at the point of contact considered. Methods of doing this have been discussed by Willis,* Rankine,† and others. Here it will be sufficient to note that the teeth of such wheels will not be of uniform section throughout their length.

FIG. 205.

In the comparatively narrow hyperboloidal wheels generally used there is but little variation in the form of the tooth in passing from one end of the wheel to the other. An approximately correct form of tooth may be determined for such wheels in the same way as for screw-wheels.

In Fig. 204 for example, we may imagine the two hyperboloids cut by a plane parallel to RYS and perpendicular

* Principles of Mechanism, p. 151. † Machinery and Millwork, p. 146.

to XOY. When the resulting sections are drawn out their circles of curvature may be approximately found, and the tooth-forms designed in the ordinary manner, remembering that the circumferential pitch, and therefore also the normal pitch, increases as we pass from the gorge to the ends of the wheels. It will be seen that this method is practically the same as that adopted in the case of ordinary bevel-wheels (see § 98), and is equivalent to drawing out the teeth on the development of the cones previously mentioned.

The subject of hyperboloidal wheels is treated at considerable length in MacCord's "Mechanical Movements," to which work the reader is referred for further information.

CHAPTER XII.

SPHERIC MOTION.

96. Spheric Motion in General. — Spheric motion has already been defined in § 6, and it has been explained that in such motion any given point in the moving body remains on the surface of a sphere described about a certain fixed point as centre. Two bodies having relative spheric motion will therefore have this point as a common centre.

We can study the relative motion of two or more such bodies by imagining that they are cut by a sphere described about the common point as centre, and we can then consider the movement of these spheric sections exactly as we considered the motion of the plane sections or projections of bodies having plane motion. Plane motion may indeed be looked upon as a particular case of spheric motion where the radius of the sphere is infinitely great.

We may therefore suppose that propositions proved with regard to plane motion will hold good, with certain necessary modifications, with regard to spheric motion also. It will be convenient, first of all, to consider the motion of a spheric figure on the sphere of motion, just as we considered in § 5 the motion of a plane figure in the plane of motion. The position of the spheric figure will of course be defined if we know the position of two of its points.

In Fig. 206 (a) a figure on the surface of a sphere LMN has the positions of two of its points (A and B) defined. Let the figure, which represents a body having spheric motion, be moved from a position AB to a new position, A_1B_1; the movement being executed in a very small period

of time, and being therefore an exceedingly small displacement. The paths of the points A and B will then practically coincide with portions of great circles passing through A and A_1, and B and B_1, respectively. Now let arcs of great circles be drawn passing through L and M, the middle points of AA_1 and BB_1; let the planes of these great circles be respectively perpendicular to those of the great circles ALA_1 and BMB_1, and let them intersect at N. Draw ON passing

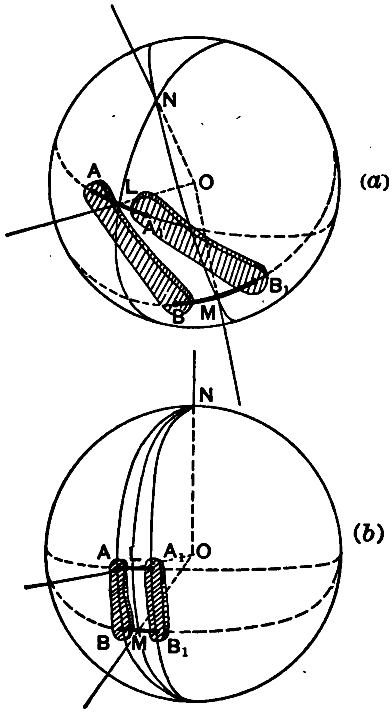

FIG. 206.

through O, the centre of the sphere. It is then evident that the actual small displacement of the body AB is the same as if it had undergone a rotation about the axis ON, for N is the point on the surface of the sphere at which AA_1 and BB_1 subtend equal spherical angles. This follows from the

fact that the spherical triangles ANB and A_1NB_1 are equal in all respects.

It may happen that L and M both lie on the same great circle, as in Fig. 206 (b), in which case our construction fails. The point N is now to be taken at the intersection of the great circles AB and A_1B_1, and it is evident, as before, that the angle subtended at N by the arcs AA_1 and BB_1 is the same, and that ON is the axis of rotation. Any actual motion of AB on the surface of the sphere may be considered as being made up of a series of infinitely small displacements, to each of which there corresponds one position of the axis ON. ON is therefore the *virtual axis* of the motion of AB with regard to the sphere. The reader should compare the foregoing argument with that in § 5 applying to plane motion.

We may call the surface described by ON in the sphere the axode of AB with respect to the sphere. Two bodies, a and b, having relative spheric motion will of course have a pair of such axodes; the axode of a being imagined as being described in the body b and *vice versa*, just as in the case of plane motion: and, further, this relative motion may be represented by the rolling together of such axodes.

Perhaps an example may make this clearer. Fig. 207 represents the pitch surfaces of a pair of bevel-wheels; their axes intersecting at O. These wheels are intended to transmit angular motion uniformly between shafts whose axes intersect, and their motion will evidently be exactly the same as that of a pair of circular cones of corresponding shape rolling together without slipping, and having a common apex at O. A pair of such cones, a and b, and their frame, c, will have relative spheric motion about the point O. The lines OA_{ac} and OA_{bc} are of course the axes of a and b with respect to the frame c; the line OA_{ab} along which the cones are in contact is the virtual axis of a with respect to b, and the surface of each cone is therefore the axode of the other. (Compare the relation between the pitch surfaces in spur-gearing.)

It is evident that in Fig. 207 the three virtual axes of the three moving bodies a, b, and c are in one plane; we proceed to show that this is true of any three bodies having relative spheric motion. The figure already used to illus-

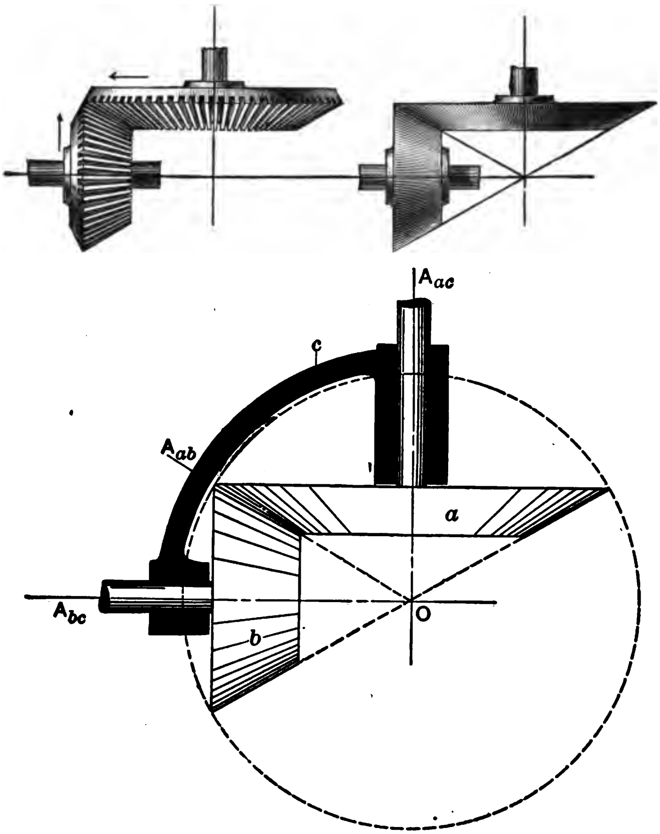

FIG. 207.

trate the corresponding proposition for plane motion is repeated here.

Let the bodies be a, b, and c; there must be some point in space, O, which is common to the three bodies, and through which, therefore, their three virtual axes must always pass. Let the paper (in Fig. 208) represent the pro-

jection of part of a spherical surface, and let O_{ab}, O_{ac}, and O_{bc} represent the traces on this surface of the virtual axes OA_{ab}, OA_{ac}, and OA_{bc}, respectively. Then, following the reasoning of § 5, we may say that the line OA_{ac} belongs, for the instant considered, to both a and c. As a line

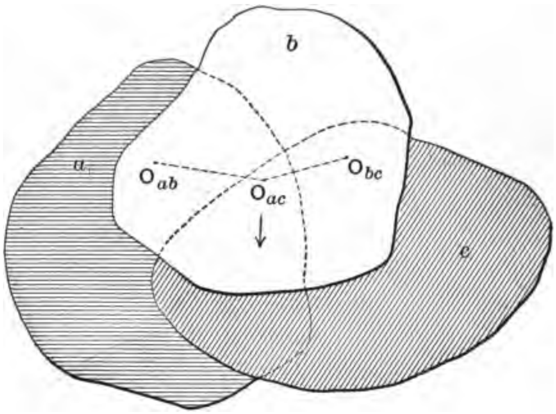

FIG. 208.

in a it is turning relatively to b about the line OA_{ab}, and is therefore moving in a direction perpendicular to the plane containing OA_{ac} and OA_{ab}. As a line in c it must similarly be moving in a plane perpendicular to the plane containing OA_{ac} and OA_{bc}. The line OA_{ac} is therefore moving in a plane normal to each of two planes which contain it, and these two planes must coincide. The three lines OA_{ab} OA_{ac}, and OA_{bc} thus lie in one plane.

97. Spheric Mechanisms having Lower Pairing. The Conic Quadric Crank-chain.—It is not difficult to devise mechanisms corresponding to the plane mechanisms of Chapter III and IV, but having spheric motion of the various links. To do this it is only necessary to arrange that the axes of all turning pairs meet in a point instead of being parallel, and that the lines of motion of sliding pairs follow great circles on the surface of the sphere of motion.

The axodes of the links of such mechanisms will, as we

have seen, be conical surfaces (not necessarily circular cones), and the mechanisms are therefore called by Reuleaux *conic chains*. A model representing a conic quadric crank-chain is shown in Fig. 209, and it may be remarked that, as in the case of plane mechanisms, the actual form of the links is unimportant from a kinematic point of view so long as the axes of the elements are in the correct position and the links do not foul one another during motion. The reader should compare the chain of Fig. 209 with that of Fig. 40.

In studying conic mechanisms we may note that, instead of considering the actual lengths of the various links, we have

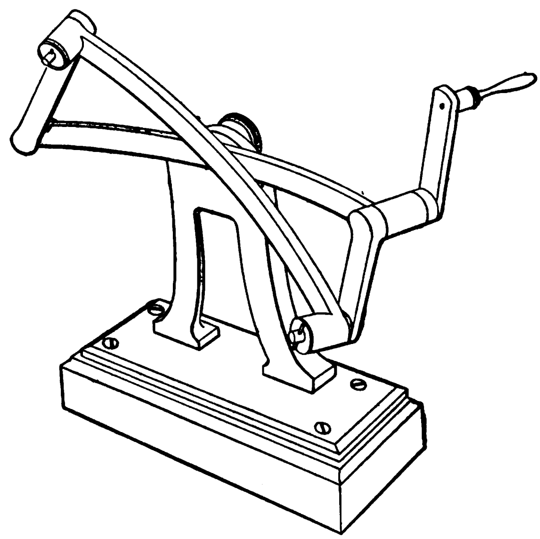

FIG. 209.

now to deal with the angles subtended by those links at the centre of the sphere. The relation between a plane mechanism and the corresponding spheric chain is in this respect like the relation between a plane triangle and a spherical one. In order to connect the elements of two turning pairs making an angle α with one another, we can thus use either a link subtending the angle α or one subtending $180° - \alpha$

Such a substitution may of course totally alter the appearance of the mechanism, but will produce no kinematic change.

The reader will recall the way in which we imagined the plane or cylindric quadric crank-chain of Fig. 60 to have one of its pairs transformed from a turning to a sliding pair. This change involved the expansion of the elements of the turning pair *cd* until the radius of their working surfaces became infinitely great. In Fig. 210 is illustrated the corresponding alteration by which a conic quadric crank-chain becomes a conic slider-crank chain. The chain of Fig. 210 (*b*) is a conic lever-crank chain, in which the angle of the links *a* and *b* is in each case 90°. It will be evident that by increasing the radius of the turning pair *ab* until it is equal to the radius of the sphere we get the chain of Fig. 210 (*c*), having exactly the same relative motions as before, but having as the link *b* a block sliding in a groove formed in *a* and following a great circle on the sphere. We may term this mechanism a spheric or conic slider-crank. The right-angle links in the conic chain correspond to the infinite links in the plane mechanism. The sketch Fig. 210 (*a*) represents a conic quadric crank-chain in which $b = 45°$. If a somewhat similar transformation were carried out in this case we should obtain a crossed conic slider-crank chain, the groove on the surface of the sphere following the trace of the axis OA_{bc}, which in this case is not a great circle and no longer passes through A_{ad}.

As an exercise the reader should endeavor to devise for himself spheric mechanisms corresponding to other simple plane mechanisms. Comparatively few conic crank-trains have found application in practice,* and in general their industrial importance is not very great. We shall proceed to follow somewhat more closely the action of the conic quadric crank-chain, which is utilized in the form of the

* For a discussion of one of these, the Tower Spherical Engine, a conic chamber quadric crank-chain, see Kennedy, Mechanics of Machinery. § 65.

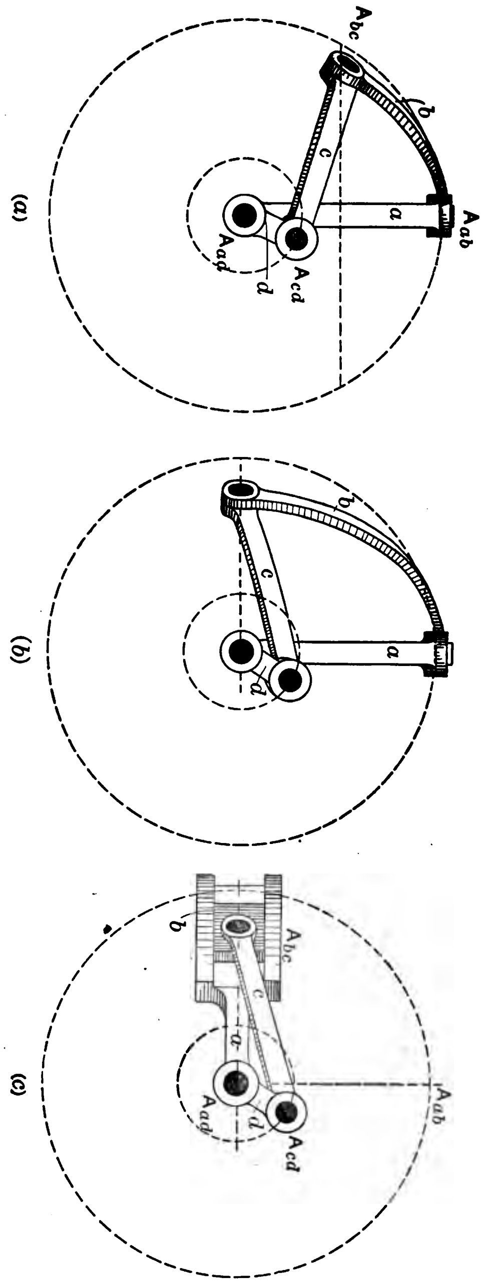

FIG. 210.

well-known Hooke's or Universal Joint, for connecting shafts whose axes are not parallel and meet in a point.

One arrangement of this mechanism is shown in Fig.

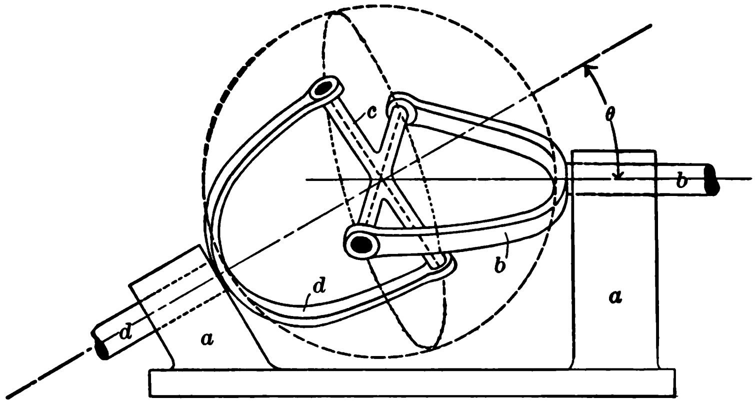

FIG. 211*a*.

211*a*, together with a diagram, Fig. 211*b*, showing the various links drawn on the surface of a sphere, after the fashion

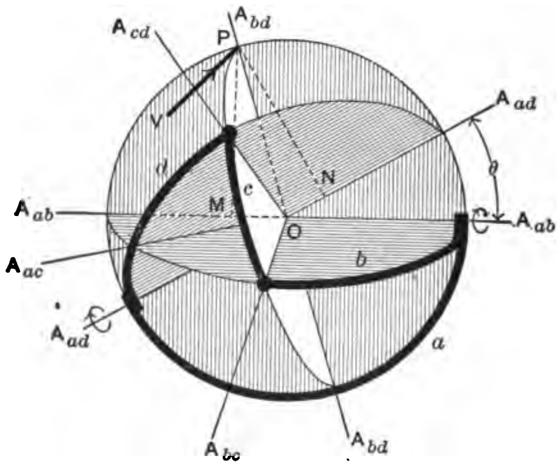

FIG. 211*b*.

of Fig. 210 (*a*). The plane of the paper is supposed to be that of the axes OA_{ab} and OA_{ac}, *b* and *d* being the links corresponding to the two shafts, while *a* is the fixed link, and *c*

connects b and d. We wish to find, for any given position of the mechanism, the angular velocity ratio of b and d. The links b, c, and d each subtend an angle of 90° at the centre of the sphere, while a subtends an angle equal to 180° – (angle between axes of shafts).

It may be noted that while b and d will obviously perform successive quarter-revolutions in equal times, their angular velocities are not necessarily equal in any given position. The angular velocities will in fact be equal at only four points in each revolution.

Utilizing the proposition of § 96, we can easily find the virtual axes of the mechanism of Figs. 211*a* and 211*b*. Thus OA_{bd} must lie in the plane containing OA_{ab} and OA_{ad}; it must also lie in that containing OA_{cd} and OA_{bc}. Similarly the point A_{ac} is found at the intersection of the great circles passing through $A_{cd}A_{ad}$ and $A_{ab}A_{bc}$. In the case of spheric motion it is important to remember that the virtual axis of two moving bodies is a line which is common for the instant to the two bodies, and which has the same angular velocity whether it is regarded as belonging to one or to the other. This statement corresponds in the case of plane motion to the definition of a virtual centre as that point in the projection of the two bodies on the plane of motion which is for the instant common to both projections, and has the same linear motion whether it is considered as a point in one body or as a point in the other.

Having found the virtual axes of the mechanism. the relative angular velocities of the links can be determined graphically. In Fig. 211*b*, for example, suppose that V is the linear velocity of the point $P(A_{bd})$ in a direction tangential to the great circle $A_{bc}A_{cd}$ and normal to the plane of the paper. The angular velocity of P (considered as a point in b) about the axis OA_{ab} is of course $\frac{V}{PM}$, and this must be equal to the angular velocity of the link b with respect to the fixed link a. Similarly, since P is also a point

in d, the angular velocity of d must be $\frac{V}{PN}$, where PM and PN are the perpendiculars dropped from P on to OA_{ab} and OA_{ad} respectively. Thus it follows that

$$\frac{\omega_{ba}}{\omega_{da}}=\frac{PN}{PM}.$$

It will be seen that the determination of the angular velocity ratio in this way involves the finding of the axis OA_{bd}, which can only be done by drawing the plane projection of the whole mechanism in each position for which the velocity ratio is required. Since each such projection generally involves drawing three ellipses, the process is not very convenient, except in the cases where the plane of the link b makes an angle of 90° with, or coincides in position with, the plane of the link a. The latter plane of course is the plane which contains OA_{ab} and OA_{ad}.

The positions mentioned are shown in Fig. 212. Let θ be the angle between the axes of b and d, let α be the angle made by the plane of b with a plane normal to the plane of the paper and containing OA_{ab}, and let β be the angle between the plane of d and that of the paper. As before, the plane of the paper contains OA_{ab} and OA_{ad}. It is then evident from the figure that when $\alpha=0$, $\beta=0$, and

$$\frac{\omega_{ba}}{\omega_{da}}=\frac{PN}{PM}=\frac{1}{\cos\theta}.$$

Similarly when $\alpha=\frac{\pi}{2}$ and $\beta=\frac{\pi}{2}$, we have

$$\frac{\omega_{ba}}{\omega_{aa}}=\frac{PN}{PM}=\cos\theta.$$

The value of the angular velocity ratio may also be found in another way. Fig. 213 shows three views of the mechanism, namely, a projection on the plane containing OA_{ab} and OA_{ad}, a projection on a plane perpendicular to OA_{ad}, and one on a plane perpendicular to OA_{ab}. In the

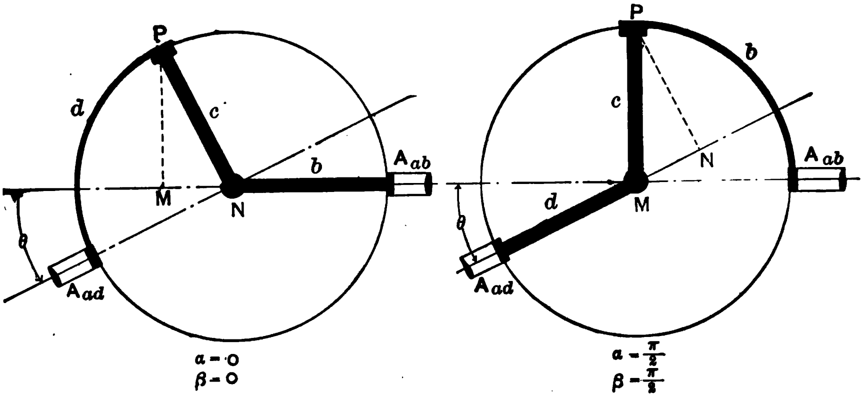

FIG. 212.

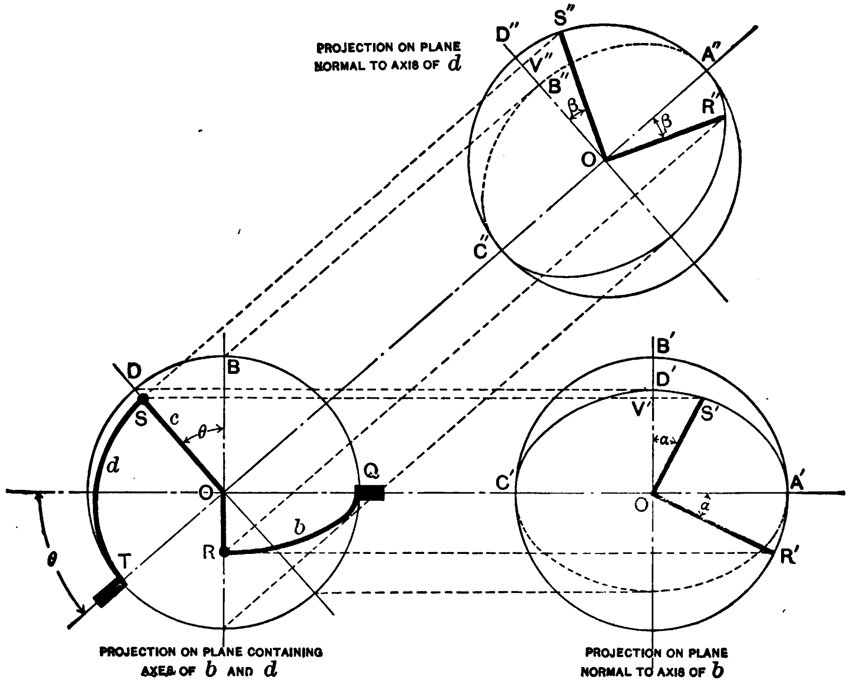

FIG. 213.

latter view the ellipse $A'D'C'$ represents the projection of the path of the point S (the join of the links c and d), while the circle $A'B'C'$ represents the projection of the path of R, the join of c and b. If now we draw the angle $A'OR'=\alpha$, so that OR' is the projection of the link b, the corresponding projection of the line OS will be OS'. The real angle ROS is of course a right angle, being equal to the angle subtended by the link c; and since the line OR' lies in the plane of the paper, the projection OS' must, by a well-known principle of projection, be at right angles to OR'. Hence it follows that the angle $B'OS'=\alpha=AOR'$. Similarly it may be shown that in the other view $D''OS''=\beta=A''OR''$. It is plain from the figure that $S'V'=S''V''=$ perpendicular distance of S from plane of axes. Also $OV'=OS\cos\theta$. Thus

$$\tan\beta=\frac{S''V''}{OV''}=\frac{S'V'}{OS}=\frac{S'V'}{OV'}\cos\theta$$
$$=\tan\alpha\cos\theta.$$

From this expression, θ being constant, we get a relation between α and β. By differentiation with regard to time

$$\sec^2\beta\frac{d\beta}{dt}=\cos\theta\sec^2\alpha\frac{da}{dt}.$$

Therefore since $\frac{da}{dt}=\omega_{ba}$ and $\frac{d\beta}{dt}=\omega_{da}$,

$$\frac{\omega_{ba}}{\omega_{da}}=\frac{1+\tan^2\beta}{\sec^2\alpha\cos\theta}.$$

But $\tan\beta=\tan a\cos\theta$. Hence

$$\frac{\omega_{ba}}{\omega_{da}}=\frac{\cos^2\alpha}{\cos\theta}+\sin^2\alpha\cos\theta$$
$$=\frac{1-\sin^2\alpha\sin^2\theta}{\cos\theta}.$$

Similarly it may be shown that

$$\frac{\omega_{ba}}{\omega_{da}}=\frac{\cos\theta}{1-\cos^2\beta\sin^2\theta}.$$

The velocity ratio plainly has its maximum value $\left(\frac{1}{\cos\theta}\right)$ when $\alpha=\beta=0$, π, 2π, etc., and its minimum value ($\cos\theta$) when $\alpha=\beta=\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, etc.; results which agree with those previously obtained.

In order to find the positions in which the shafts have the same angular velocity we have only to put $\frac{\omega_{ba}}{\omega_{da}}=1$ and

$$1-\sin^2\alpha=\cos\theta(1-\sin^2\alpha\cos\theta)$$

or

$$\sin^2\alpha=\frac{1-\cos\theta}{1-\cos^2\theta}.$$

For example, if $\theta=30°$ and $\cos\theta=0.86602$, we have

$$\sin^2\alpha=\frac{0.13398}{0.25001}=0.53590,$$
$$\sin\alpha=\pm 0.73205.$$

The two shafts will then be moving with the same angular velocity when $\alpha=47°\ 4'$, $132°\ 36'$, $227°\ 4'$, or $312°\ 36'$; that is to say, four times in each revolution.

It is evident from the relations thus obtained that if we connect two shafts by means of an intermediate piece and two similar universal joints, as is done, for example, in the feed gear of certain milling machines, then if the shafts are parallel so that $\theta_1=-\theta_2$, we have $\alpha_1=\beta_2$, and the shafts will have uniform velocity ratio; the inequality of motion caused by the first universal joint being exactly compensated by the second.

The method of studying the action of the conic quadric crank-chain may serve as an example of the way in which other conic mechanisms having lower pairing may be treated.

98. Spheric Mechanisms having Higher Pairing. Bevel-gear.—Each of the spheric mechanisms discussed in the preceding section is the representative of a plane mechanism, the essential difference being that in a spheric mechanism

the axes meet in a point instead of being parallel, and the relative motion of the links is spheric instead of plane.

If we consider in a similar manner the change which would take place if the axes in spur gearing were made to intersect instead of being parallel, it is plain that the cylindrical pitch surfaces would become cones, whose apices would lie at the point of intersection of the axes. The toothed wheels, whose relative motion corresponds to the rolling together of such conical pitch surfaces, are known as *bevel-wheels*, and such relative motion is, of course, spheric motion, as was shown in § 96.

If we go a step further and imagine that the axes are not parallel and do not intersect, then the pitch surfaces become hyperboloids and the relative motion is screw motion, a state of things which has already been considered in Chapter XI.

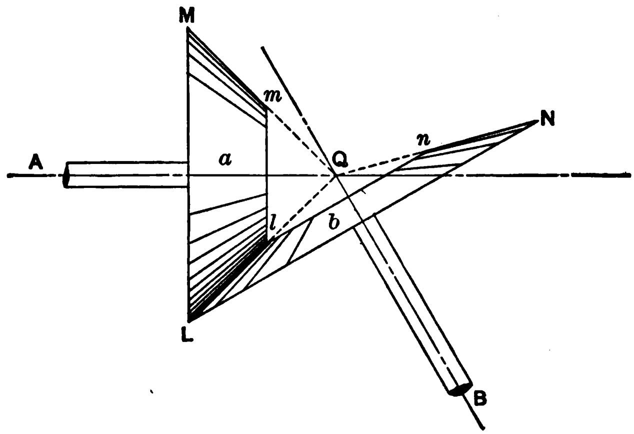

FIG. 214.

Two right circular cones, a and b, whose axes are AQ and BQ, are in contact as represented in Fig. 214. If these cones are so rotated that there is no relative slipping at one point of contact, L, and if the point Q is the common apex of both cones, then at every other point of contact there will

also be rolling, without any sliding motion. Let the plane of the paper contain the axes QA and QB and the line of contact QL; then, if there is no slipping at L, and if ω_{ac} and ω_{bc} are the angular velocities of a and b, relatively to c, the fixed link (not shown in the figure), we shall have

$$\frac{\omega_{ac}}{\omega_{bc}}=\frac{V_c}{LM}\cdot\frac{LN}{V_c},$$

where V_c is the common linear velocity of the two cones at L, measured, of course, in a direction normal to the plane of the paper. Hence

$$\frac{\omega_{ac}}{\omega_{bc}}=\frac{LN}{LM}.$$

Further, if there is to be no slipping at another point of contact, l, we must have

$$\frac{\omega_{ac}}{\omega_{bc}}=\frac{ln}{lm},$$

a relation which shows that when $\frac{ln}{lm}=\frac{LN}{LM}$ the circular sections at ln and lm will roll together if the sections at LN and LM do so. We may note that

$$\frac{\omega_{ac}}{\omega_{bc}}=\frac{\sin BQL}{\sin AQL}.$$

It is easy therefore to lay out the pitch surfaces for a pair of bevel-wheels having any desired velocity ratio. We have only to arrange two cones having a common apex, and having a line of contact such that the lengths of the perpendiculars dropped from any point on it to the axes are inversely as the angular velocities. The cones need not necessarily have external contact. Bevel-wheels having internal contact can be made; an internal bevel-wheel corresponds to an annular spur-wheel.

In practice frustra of the pitch-cones are used for the

pitch surfaces of bevel-wheels, and slipping is prevented by forming teeth on these pitch surfaces, exactly as in the case of spur-wheels, with the important difference that in the case of bevel-wheels the teeth are not of uniform section, but taper in such a fashion that they would vanish at the apex of the pitch-cone if they were continued to that point.* In Fig. 214 the virtual axes are, of course, QA, QL, and QB, QL being the virtual axis of a with regard to b. The third (fixed) link of the train is not shown.

In Chapter VII, when studying the formation of the tooth profiles for spur gearing, we considered these as being plane curves, described upon a plane normal to the axes of the wheels and (in the case of cycloidal teeth) generated by the rolling together of the describing circles and the pitch-circles. It would perhaps have been more correct if we had considered the tooth *surfaces* as being generated by the rolling together of pitch *surfaces* and describing *cylinders*. In the case of bevel gearing the corresponding problem is more difficult, because it is necessary to picture in the mind the working faces of the teeth as being described by the rolling together of conical, instead of cylindrical, surfaces. Drawings connected with bevel-gear in general will require at least two projections, as we found when considering hyperboloidal gearing.

It is possible to assume the form of the working surface of the tooth of one bevel-wheel, and then to devise the form of the corresponding tooth of another wheel gearing with the first with uniform velocity ratio. The necessary and sufficient condition for such uniformity will be that in any spheric section of the pair of teeth, drawn with the intersection of the axes as centre, a great circle drawn to cut the tooth profiles at right angles at their point of contact will pass through the virtual axis of the pair of wheels. (Compare the corresponding condition for plane motion.) It is proper to employ spherical curves as forms for the teeth of bevel-wheels, these curves being drawn in similar fashion

* See Fig. [illegible]

to the plane involute or cycloidal curves used in the case of plane motion, and we shall now discuss the way in which we may imagine involute bevel-wheel teeth profiles to be generated. In Fig. 215 the curves MSC, NTD represent the

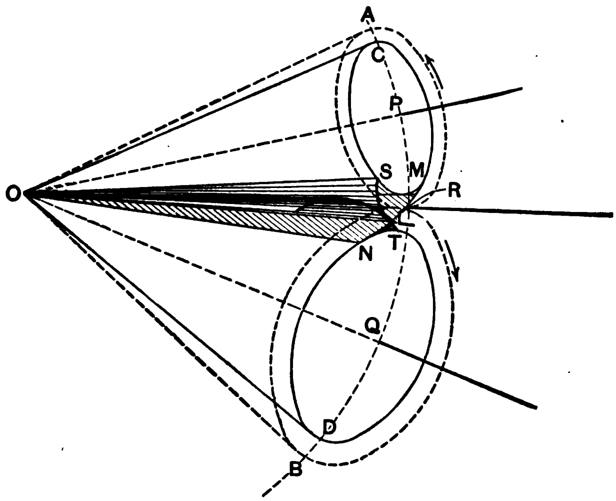

FIG. 215.

traces, on a spherical surface whose centre is O, of two right circular cones, having axes OP and OQ. The great circle $APRQB$ lies in the plane containing OP and OQ. A plane touching the two cones intersects the spherical surface in MRN, which is, of course, an arc of a great circle. If now we suppose that the plane surface OMN takes the form of a flexible sheet, it will be seen that a rotation of the two cones in the sense of the arrows would cause the sheet to wrap itself around the cone OSM and unwrap from the cone OTN, a point L on the edge of the sheet thus describing on the surface of the sphere the great circle NLM. With reference to the small circle SM, the point L will describe on the surface of the sphere the curve SL, which may be termed a spherical involute; and in the same way with regard to TN, the involute TL will be drawn, the two in-

volutes, of course, always touching at L. The line OL will thus describe on the cone OSM the ruled surface OSL, and the somewhat similar surface OTL will be generated by the relative motion of OL and the cone OTN. These surfaces will roll and slide together, always being in contact along such a line as OL, if the cones rotate with uniform velocity ratio; and they can therefore be used as the working faces of the teeth of a pair of bevel-wheels whose axes are OP and OQ. The pitch surfaces of these wheels will be the cones OAR, OBR, which are indicated by dotted lines in the figure and touch along the line OR. A plane drawn through OL perpendicular to the plane OMN would be the common tangent plane to the tooth surfaces through their line of contact. The reader should compare this discussion with that of § 66, noting the modifications rendered necessary in adapting the reasoning to the case of spheric motion. He should also endeavor to work out for himself the method of forming cycloidal teeth for bevel-wheels by a method corresponding to that used in § 67 for spur-wheels, remembering that the figure must be supposed to be drawn upon the surface of a sphere instead of upon a plane surface.

In practice it is not convenient to use spherical surfaces for drawing; in setting out the teeth of bevel-wheels it is therefore necessary to adopt a somewhat different method (due to Tredgold) which gives results closely approximating to the truth.

In Fig. 216, OAR, OBR represent the pitch surfaces of a pair of bevel-wheels, projected on a plane containing the axes OP, OQ. The arc $APRQB$ is the trace of a spherical surface drawn with centre O, on which surface the outlines of the teeth should properly be described. For this surface we substitute the developable surfaces of the two cones XAR, YBR which touch the spherical surface along the pitch-circles AR, BR of the bevel-wheels. It is evident that these cones are in fact the same as the cones mentioned (in the case of hyperboloidal wheels) in § 95,

and their curved surfaces are normal to the pitch surfaces of the wheels. The arcs RV and RW, drawn with X and Y as centres respectively, are in fact the developments of

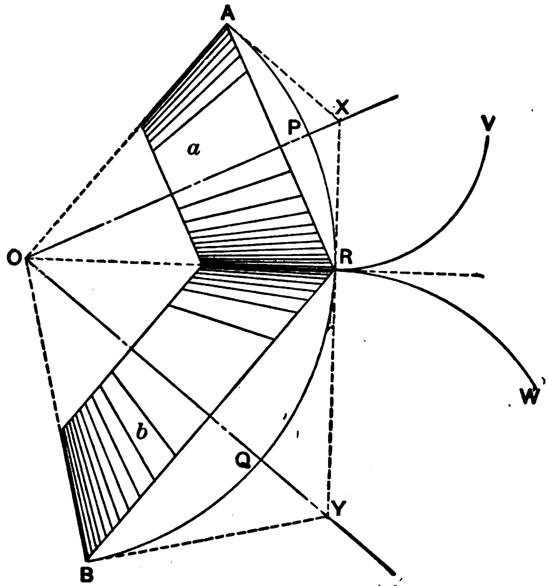

FIG. 216.

portions of the pitch-circles AR, BR, and it is on these lines as pitch-circles that the tooth profiles are to be drawn.* No considerable error is introduced by this construction so long as the bevel-wheel has more than 24 teeth.

In recent years a number of machines have been designed for the purpose of cutting bevel-wheel teeth †; one of the most important and interesting from a kinematic point of view is that devised by Hugo Bilgram ‡ for cutting teeth of the involute form. In this machine the correct form of

* See Unwin, Machine Design, Vol. I. § 208.

† Trans. Am. Soc. M. E., Vol. XXII. p. 672.

‡ See *Engineering*, Vol. XL. p. 21; also Journal of the Franklin Inst., Aug., 1886.

tooth is generated by the relative motion of a V-shaped cutter, representing the involute tooth of an imaginary plane bevel-wheel or crown-wheel, and of the wheel blank. The latter is rotated, during the formation of each face of each tooth, about its proper virtual axis. The meaning of the term plane bevel-wheel will be seen from Fig. 217.

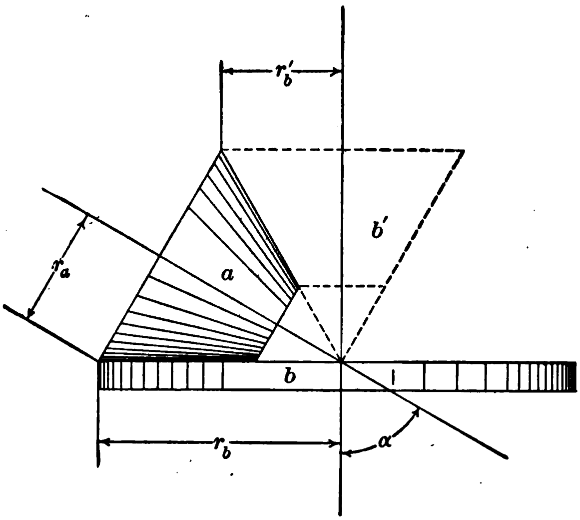

FIG. 217.

In this figure the pitch surface of a is a cone whose vertical angle is $2\left(\frac{\pi}{2}-\alpha\right)$, where α is the angle between the axes. The pitch surface of b then has a vertical angle of 180° and is a plane. From the figure we see that

$$\frac{\omega_b}{\omega_a}=\frac{r_a}{r_b}=\cos\alpha.$$

With the same angle between the axes we might also have a wheel b' of vertical angle $2\left(\alpha-\left[\frac{\pi}{2}-\alpha\right]\right)$ gearing with a. In this case we should have a velocity ratio

$$\frac{\omega_{b'}}{\omega_a} = \frac{r_a}{r_{b'}} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\sin\left(2\alpha - \frac{\pi}{2}\right)}$$

$$= \frac{\cos\alpha}{1 - 2\cos^2\alpha}$$

If in Fig. 217 we suppose the pitch surface of the plane wheel b to be fixed, and the wheel a to be rolled upon it, it is plain that a cutting-tool reciprocating along a diameter of the pitch surface of b will in successive cuts, if properly adjusted, form a tooth space in the wheel blank corresponding to a. It is on this principle that the action of the Bilgram machine depends.

In another class of bevel-gear-cutting machines we may place those using a master gear or template. The mode of operation of the Rice gear-cutting machine, which is of this type, is illustrated in a diagrammatic fashion in Fig. 218; the details of the actual machine being somewhat differently arranged. The blank A, from which the bevel-wheel is to be cut, has the tooth spaces roughly gashed out, and is mounted on a shaft to which is secured a template or master wheel B, having teeth of the correct profile formed upon it. There may be only one of these profiles, as in the sketch, or B may take the form of a complete wheel. The shaft carrying A and B can rotate about an axis OX, with reference to the frame D, which in its turn can be rotated about a vertical axis OY. The actual motion of the blank may therefore be any rotation compounded of movements about the axes OX and OY. The fixed base of the machine, E, carries an arm E_1, in connection with which a rotating cutter C and a guide-plate F are so arranged that the face of the cutter and the face of the guide-plate lie in a plane containing the axis OY. The cutter having slightly entered the gash cut for the tooth space, the frame D is rotated about OY, and if the tooth form is kept in contact with the

plane face of F, it is evident that the relative motion of F and B will be copied on a reduced scale by C and A, and the cutter will therefore form one side of one tooth of the blank wheel. With reference to the fixed link E, the blank and template are at any instant really rotating about some such axis as OR. In order to cut the opposite side of the tooth, the stop F must be moved parallel to itself by an amount

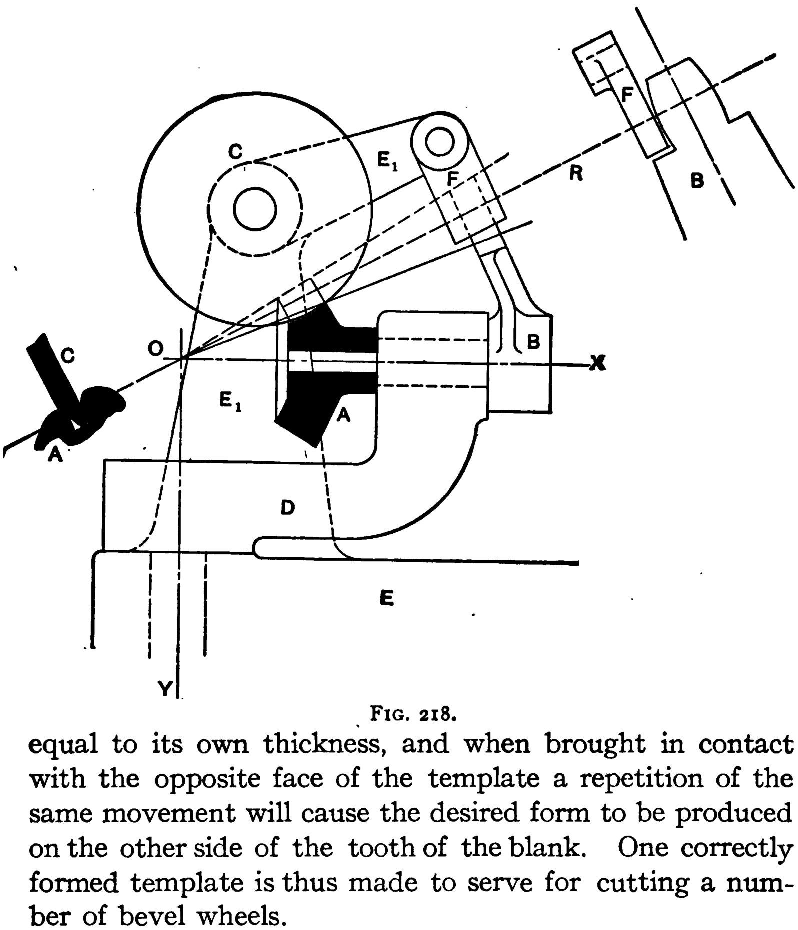

FIG. 218.

equal to its own thickness, and when brought in contact with the opposite face of the template a repetition of the same movement will cause the desired form to be produced on the other side of the tooth of the blank. One correctly formed template is thus made to serve for cutting a number of bevel wheels.

It is possible to construct bevel-wheels with spirally formed teeth, as in Fig. 219. These wheels differ from the

skew-bevel wheels of Fig. 205 in that the axes of the wheels intersect, so that the pitch surfaces are of course cones, and not hyperboloids. The teeth are no longer straight, but follow helical curves traced on the conical surfaces. Such

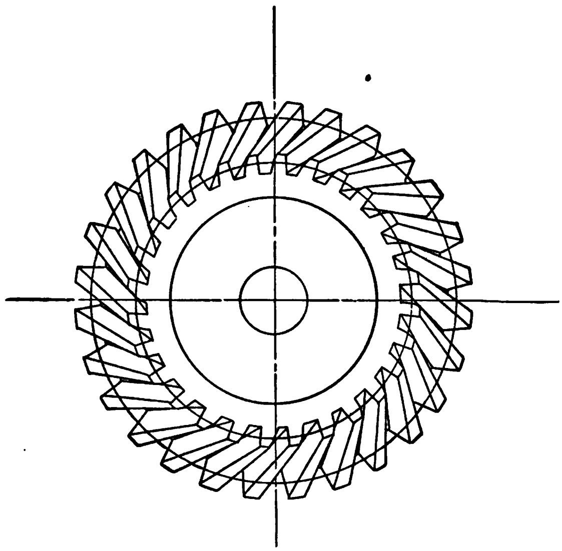

FIG. 219.

wheels correspond in fact to screw-wheels, and have recently met with considerable favor on the continent of Europe, several French and German gear-cutting machines being specially designed for producing them. The teeth of spiral bevel-wheels usually follow a conical helix of constant pitch, the projection on the base of the cone being an Archimedean spiral.

The velocity ratios of bevel-gear mechanisms can be determined by aid of the principles discussed in §§ 68 and 69, and these gears may be arranged so as to correspond with the various compound or epicyclic trains described in a previous chapter; they may include annular wheels, and the general methods of determining their velocity ratios are the same as those employed in the cases

previously discussed. We shall take a few examples which will make this clear.

In Fig. 220 is represented a well-known gear* which finds wide application. It consists of four bevel-wheels a, b, c, d, the frame or fixed link being e; of these wheels a and b are of the same size, and c and d are also equal. The axes of the wheels intersect at right angles as shown.

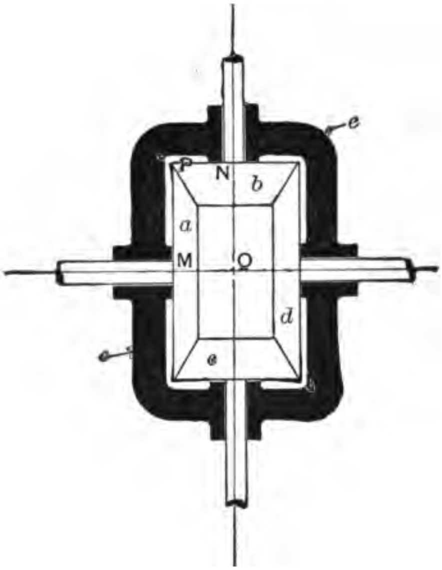

FIG. 220.

If we imagine the frame e to be the fixed link, it is obvious that the following relations will hold:

$$\omega_{ae} = -\omega_{de} = \omega_{ed},$$

$$\omega_{be} = -\omega_{ce} = \omega_{ec}.$$

We have also

$$\frac{\omega_{ae}}{\omega_{be}} = \frac{PN}{PM} = -\frac{\omega_{de}}{\omega_{be}},$$

and $\omega_{ae} + \omega_{ed} = \omega_{ad}$.

Hence $\omega_{ae} = \frac{1}{2}\omega_{ad}$ and $\omega_{ea} = \frac{1}{2}\omega_{da}$.

Similarly, $\omega_{ce} = \frac{1}{2}\omega_{cb}$ and $\omega_{ec} = \frac{1}{2}\omega_{bc}$.

* The so-called "Differential" Bevel-gear.

We thus see that if a be the fixed link, e will revolve around the axis MO with one half the angular velocity of d, and in the same sense.

A compound reverted bevel-gear train is shown in Fig. 221; in this particular case the axes do not intersect at right angles. In order to find the angular velocity ratio of a and c we have

$$\frac{\omega_{ad}}{\omega_{bd}} = -\frac{PN}{PM} \quad \text{and} \quad \frac{\omega_{bd}}{\omega_{cd}} = -\frac{QS}{QR};$$

thus
$$\frac{\omega_{ad}}{\omega_{cd}} = \frac{PN \,.\, QS}{PM \,.\, QR}$$
$$= \frac{TR \,.\, QS}{TV \,.\, QR}.$$

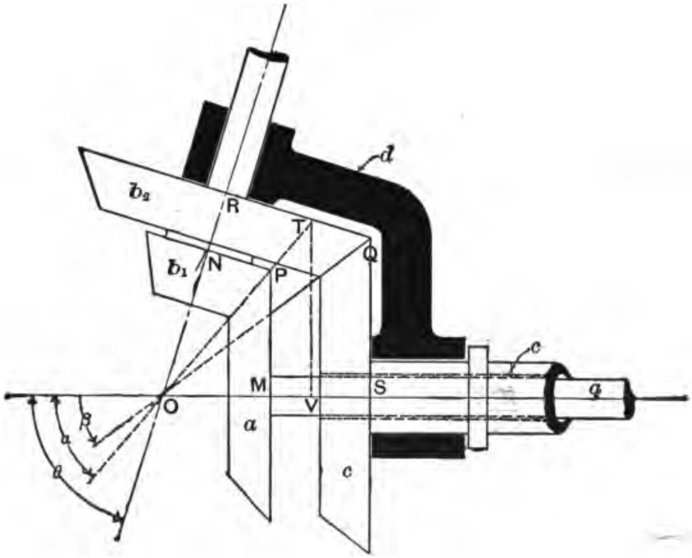

FIG. 221.

If now we denote the angles TOS, QOS, and ROS by α, β, and θ respectively, it is evident that

$$\frac{\omega_{ad}}{\omega_{cd}} = \frac{\sin(\theta-\alpha)\sin\beta}{\sin(\theta-\beta)\sin\alpha}.$$

Let n be the numerical value of this expression. If now we imagine that c makes one revolution, d being fixed, a will make n revolutions in the same sense. Let the whole mechanism receive one revolution in the negative sense about the axis OS, so that c is brought to rest, and d makes one revolution in the negative sense. While d makes one revolution the link a will now be making $n-1$ revolutions in the sense opposite to that of the rotation of d. We then find that

$$\frac{\omega_{ac}}{\omega_{dc}}=n-1=\frac{\cot\alpha-\cot\beta}{\cot\beta-\cot\theta}.$$

It should be noted that if the axes are at right angles,

$$n=\frac{\omega_{ad}}{\omega_{cd}}=+\frac{TR}{QR}=\frac{\cot\alpha}{\cot\beta},$$

and

$$\frac{\omega_{ac}}{\omega_{dc}}=n-1=\frac{\sin(\beta-\alpha)}{\sin\alpha\cos\beta}.$$

Fig. 222 shows the way in which the bevel-gear train of

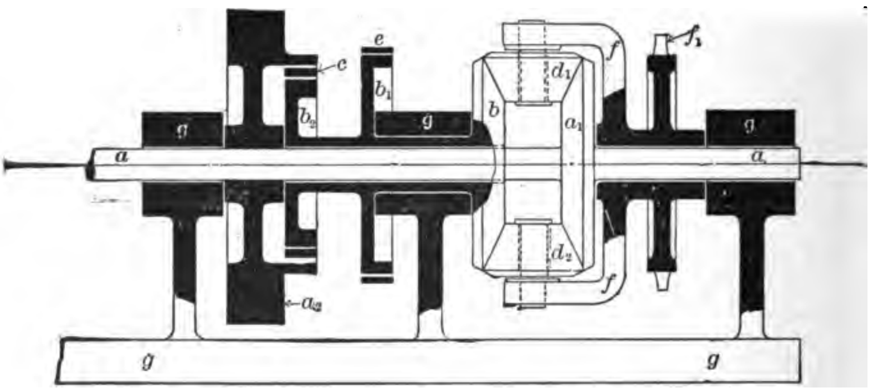

FIG. 222.

Fig. 220 is employed as a two-speed gear for transmitting power from the engine of an automobile to the driving-wheels. The engine-shaft a is rigidly connected to a bevel-

wheel a_1. A second bevel-wheel, b, similar to a_1 runs freely on the shaft, but can be held at pleasure by the application of a band-brake e to its brake-drum b_1. The wheel b_2 is rigidly connected to b and can be driven by the engine fly-wheel a_2 by means of a friction-clutch c, the details of which are not shown.

A pair of smaller bevel-wheels d_1, d_2 are carried by a frame f, which runs freely on the engine-shaft, and has upon it the sprocket-wheel f_1, which gears with the pitch-chain that transmits the motion to the driving-wheels (see Fig. 167). The fixed piece or frame of the mechanism is g.

The gear can be worked under the following four conditions:

(1) Engine running freely. The clutch c is disconnected, f remains stationary, the brake e is not applied, and b rotates with angular velocity equal to that of the engine-shaft, but in the reverse sense.

(2) Engine drives sprocket-wheel at low speed. In this case b_1 is held by the brake e, the clutch c is not in gear, and f rotates in the same sense as a_1, but with only half its angular velocity.

(3) Engine drives sprocket-wheel at high speed. The clutch c now connects a_2 and b_2, the brake e is not applied, a, b, and f therefore all revolve together with the same speed, and the bevel-wheels have no relative motion at all.

(4) Carriage is stopped by applying the brake e, the clutch c being in gear, thus connecting b and a.

99. Roller Bearings Involving Spheric Motion.—We have seen that the various links in bevel-gear mechanisms have relative spheric motion. Many roller and ball bearings, which may be regarded kinematically as augmented turning pairs, present examples of similar relative movement, which will next be discussed. The arrangement shown in Fig. 223 is often employed as a thrust bearing for a shaft or pivot on which a longitudinal pressure is exerted. The plates or flanges on which the pressure is taken are

separated by a number of conical rollers, cages or frames being usually provided so as to prevent the rollers from getting out of position. In such a case, if we know ω_{ab} (the angular velocity of the flange or collar a with regard to the fixed link b), the angular velocity of the rollers can be exhibited in a very simple manner. In Fig. 223, with reference to the fixed link the roller c is turning about the instantaneous axis OB, while OA is its instantaneous axis with

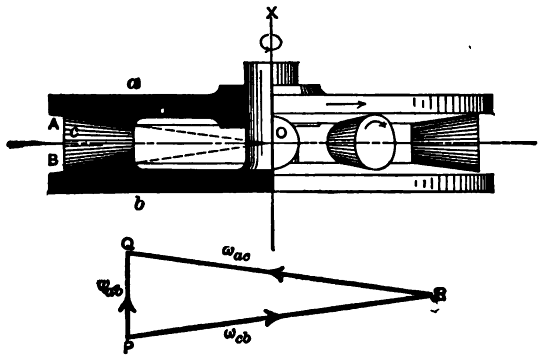

FIG. 223.

relation to a. OX is the axis about which a is turning relatively to b. The real motion of c, relatively to b, may therefore be considered as being the resultant of the motions of c relatively to a, and of a relatively to b.

From a pole P we draw the vector PQ, representing ω_{ab}, according to the convention of § 16. The line PR is then drawn parallel to BO, the instantaneous axis of c with regard to b, and it meets a line QR drawn parallel to AO. Plainly c revolves relatively to b in a sense shown by the arrow-head on PR, and the arrow-head on RQ gives the sense in which a turns with regard to c. PQR is then a triangle of angular velocities in which

$$PR = \omega_{cb},$$
$$RQ = \omega_{ac},$$

and PQ, the resultant of QR and RP, must represent the resultant of ω_{ac} and ω_{cb}, namely, ω_{ab}.

The motion of the roller c with respect to the links with which it pairs is evidently the same as that of a bevel-wheel, and the whole really forms a spheric mechanism.

Next suppose that the rollers are cylindrical, as in Fig. 224, and, as before, let OA, OB be the virtual axes of a and c and of c and b. The motion of the roller c relatively to a or to b, may now be shown to consist of *rolling* about an axis parallel to OC, combined with a *spin* about an axis parallel to AB. If we draw the triangle of velocities, as before, and proceed to resolve ω_{cb} into two components, one (PS) parallel to the surfaces in contact, and the other

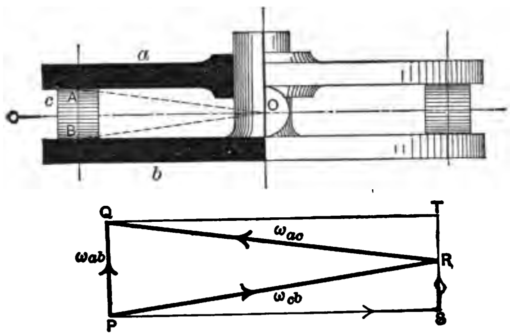

FIG. 224.

(SR) normal to those surfaces, we may evidently consider the real motion of c relatively to b as being due to a rolling PS combined with a spinning SR. The latter line then represents the angular velocity with which c is being twisted about a vertical axis in order to compel it to travel round in a circular path. In the same way RT gives the spinning velocity of a relatively to c. In Fig. 223 the form of the rollers is such that ω_{cb} has no component normal to the surface of contact; hence in that case there is, as has already been seen, no spinning, but a relative motion of rolling only.

In Fig. 224 the line ST gives the total spinning at A and at B; this is evidently equal to PQ, the relative angular

velocity of a and b about the axis OX. The condition for rolling without spinning is that the instantaneous axis of the two bodies shall lie in the tangent plane at the point of contact (§ 6); this is the case in Fig. 223, but not in Fig. 224.

Roller-bearings are often used for the journals of shafts; but the rollers are then parallel, instead of conical, and their motion is not spheric but plane.

100 Ball-bearings. — When balls are substituted for rollers, the motion is in general spheric, and Fig. 225 shows diagrammatically the arrangement of a number of typical ball-bearings, as applied to shaft-journals (a, b, c) and to a footstep- or thrust-bearing (d). It will be seen that these bearings may be classified into two-, three-, or four-point bearings, according to the number of points of contact between each ball and its races. The form (a) is a two-point bearing, and in practice it is necessary to give the working surfaces of the races a slight curvature in the direction of the axis of the shaft, as shown, in order to keep the balls in position. At (b) is shown a four-point bearing, and at (c) a two-point bearing, as used for the wheel or crank-axle bearing of a bicycle; (d) is a three-point thrust-bearing. Both the balls and their races are made of the hardest suitable material, so as to reduce as far as possible the surface of contact. If the geometrical forms of the parts were exact, and if the material were perfectly rigid, contact would take place at points only, and any loss due to spinning friction at such points of contact would be negligible. Actually, however, this loss is quite appreciable, especially where the balls are heavily loaded, and, other things being equal, the best ball-bearing will be one in which this spinning of the balls about axes normal to their surfaces of contact is reduced to a minimum. The loss from rolling friction is not of such great importance. The method of determining the rolling and spinning velocities of the balls will now be considered for various cases.

In Fig. 226 is represented a chain consisting of a fixed cone b, and a ball c rotating in contact with b and also in

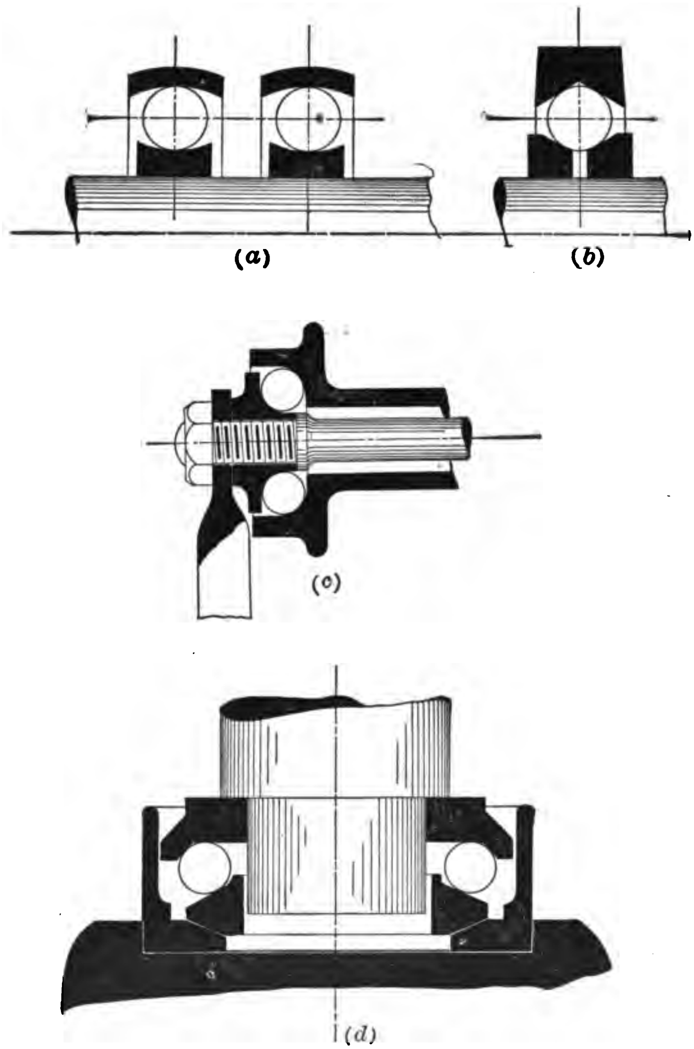

FIG. 225.

contact with the movable cone a. This movable cone

rotates relatively to b about their common axis XY. It is required to determine the motion of the ball c. The whole arrangement may be supposed to form one portion of a two-point ball-bearing, the cones a and b being the tangent cones to the curved races in such a bearing as Fig. 225 (c).

In order to see more clearly the nature of the relative movement of the ball and cones, suppose the ball to be

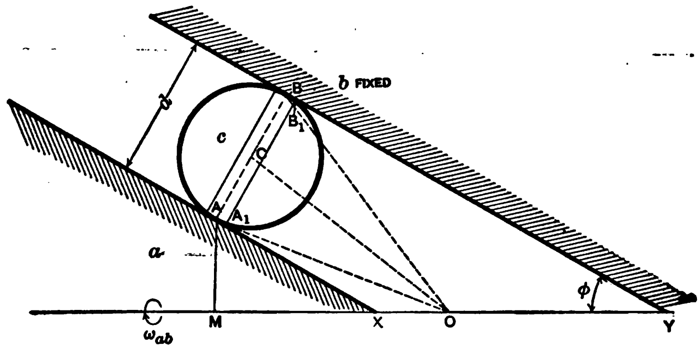

FIG. 226.

replaced by a very thin cylindrical roller of diameter d and thickness Δt, so that $AA_1 = BB_1 = \frac{\Delta t}{2}$. Let ω_{ab} be the relative angular velocity of a and b; then the linear velocity of the point A in a direction normal to the plane of the paper will be $\omega_{ab} \times AM$. The velocity of a point A_1 (on the cone) will be $\omega_{ab} \times \left(AM - \frac{\Delta t}{2}\sin\varphi\right)$, while the velocity of the point A_1 (on the roller) will be the same as that of the point A_1, namely, $\omega_{ab} \times AM$; that is, if we suppose the roller to be moving about the instantaneous axis BY, or if we suppose there is pure rolling at B. Hence A_1 on the cone and A_1 on the roller will have a relative linear velocity (normal to the plane of the paper) equal to

$$\omega_{ab} \cdot \frac{\Delta t}{2} \cdot \sin \varphi,$$

and their relative angular velocity about the axis AB will be

$$\frac{\omega_{ab}\frac{\Delta t}{2}\sin\varphi}{\frac{\Delta t}{2}} = \omega_{ab}\sin\varphi,$$

which is in fact the rate at which the ball will spin on the cone a if it rolls without spinning on the cone b. In other words, the total relative spinning of the ball with regard to the cones is $\omega_{ab}\sin\varphi$.

In general we have no right to assume that BY is the virtual axis of c and b, and that AY is the virtual axis of c and a. It is just as likely that these axes are BX and AX, in which case there would be a spin $\omega_{ab}\sin\varphi$ at B and pure rolling at A. We know that the centre of the ball travels in a circular path whose plane is perpendicular to XY, and that the three virtual axes lie in one plane. That the ball has spheric motion will be seen if we imagine that the whole bearing receives such a velocity that the centre of the ball remains at rest, while b and a rotate about XY with angular velocities of different senses. The points A and B are now moving perpendicularly to the plane of the paper, and the axis of rotation of the ball must therefore be some line lying in the plane of the paper, and cutting XY in some point O. The position of O will depend on the relative amount of rolling and spinning at A and B, and it forms the centre of the spheric motion for the whole bearing, so that the virtual axes of c and b and of c and a must pass through O. The line OC will of course be the axis of rotation of the ball, supposing the whole bearing to be moved so that C is at rest. To obtain the real motion of the ball when b is fixed, we must compound the rotation about OC with another about XY. It seems reasonable to suppose that if the surfaces of contact are equally rough, the total spin of the ball on the cones will be equally divided between A and B. The various virtual axes will then be as shown

in Fig. 227, where AO is the virtual axis of a and c, and BO is the virtual axis of c and b. The point O is found by drawing BO parallel to CX when C is the centre of the ball. The various velocities are easily found graphically, after the manner of Fig. 224. In Fig. 227 draw PQ representing ω_{ab} to

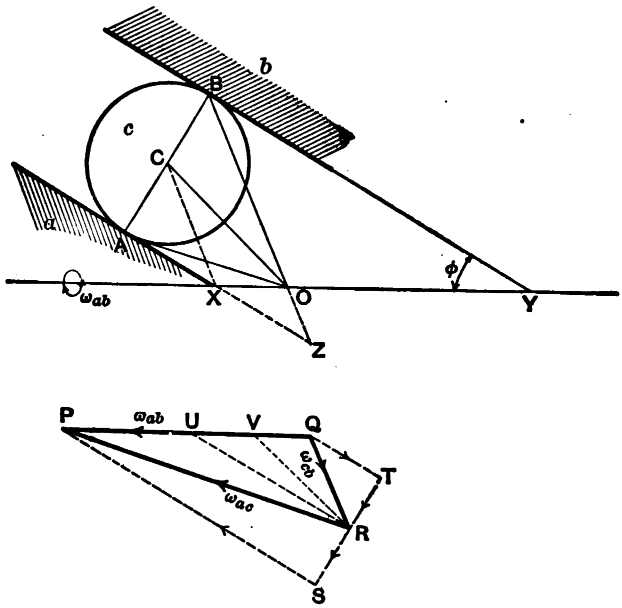

FIG. 227.

any convenient scale; then PR and QR, drawn parallel respectively to AO and BO, will represent ω_{ac} and ω_{cb}. Each of these velocities can be resolved into a spin and a roll; thus PS is the angular velocity of rolling, and RS that of spinning, at A, while QT and TR are the rolling and spinning velocities at B. We have to show that $RT = RS$. Draw RU parallel to AX or BY, and produce BO to meet AX in Z. Then the triangles AOX, RPU are similar, and the triangles ZOX, RQU are similar, so that

$$\frac{PU}{UQ} = \frac{PU}{UR} \cdot \frac{UR}{UQ}$$

$$=\frac{OX}{AX}\cdot\frac{ZX}{OX}$$
$$=\frac{ZX}{AX}.$$

But by construction $AX=ZX$, since $AC=CB$. Therefore $PU=UQ$ and $SR=RT$, so that the spin at A is equal to that at B.

It is noteworthy that ST represents the total spinning at A and B, which is readily seen to be equal to $\omega_{ab}\sin\varphi$.

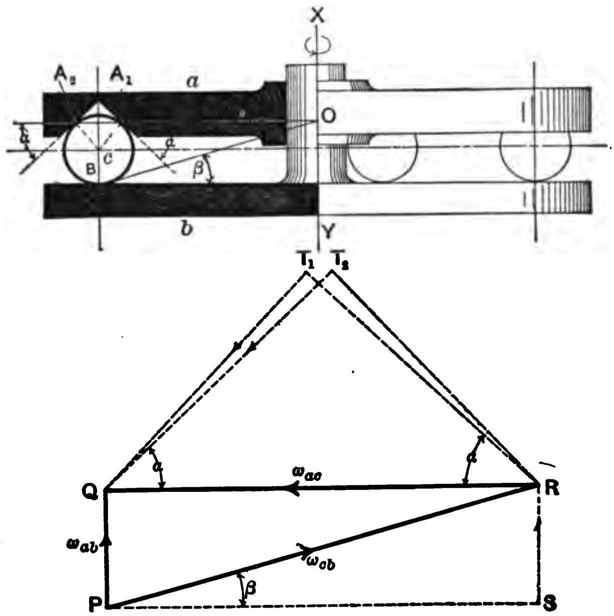

FIG. 228.

The difference of the rolling velocities at A and B is evidently $\omega_{ab}\cos\varphi$.

In order to find the angular velocity of the centre of the ball around XY, we draw RV parallel to OC. We thus resolve QR, the real angular velocity of c about the axis OB relatively to the fixed link, into QV, a rotation about XY, and VR, a rotation about OC. QV would then be the angu-

lar velocity to be given to the whole bearing so as to bring the centre of the balls to rest.

The next example is a three-point thrust-bearing of the type of Fig. 225 (*d*), as shown in Fig. 228.

In this case it is evident that the ball must roll relatively to a about a virtual axis OA_1A_2, passing through both points of contact A_1A_2, while OB must be the virtual axis of c relatively to the fixed plate b. Let α and β be the angles made by the virtual axes with the surfaces of contact, then the velocity diagram PQR is drawn as in the previous example. On resolving ω_{ac} and ω_{cb} into their component spinning and rolling velocities, as shown by the dotted lines, we find that

Spinning velocity at $A_1 = QT_1 = \omega_{ac} \sin \alpha$;
" " " $A_2 = RT_2 = \omega_{ac} \sin \alpha$;
" " " $B = RS = \omega_{cb} \sin \beta$
$= \omega_{ab}$.

We find also that

Rolling velocity at $A_1 = RT_1 = \omega_{ac} \cos \alpha$;
" " " $A_2 = RT_2 = \omega_{ac} \cos \alpha$;
" " " $B = PS = \omega_{cb} \cos \beta$
$= \omega_{ac}$.

The relative angular velocities for a three-point bearing in the more usual case where OA_1A_2 is not perpendicular to OX will be determined in a numerical example, after considering the case of a four-point bearing, which may be worked out by similar methods.

In Fig. 229 A_1A_2 B_1B_2 are the points of contact, and plainly if there is to be spheric motion on the part of c, A_1A_2 and B_1B_2 must meet at some point O on XY. If this is not the case, slipping will occur at some one or more of the points of contact, and the relative motion at these points will no longer consist simply of rolling and spinning combined. In the figure OA_1A_2 and OB_1B_2 are the virtual axes of a and c, and of b and c, respectively. Let ω_0 be the angular velocity of the centre of the ball,

so that the linear velocity of the point C in a direction normal to the plane of the paper will be $\omega_0 \times CL$. Since A is a point on OA_1A_2, its linear velocity will be $\omega_{ab} \times AM$ in the same direction. But b is the fixed link

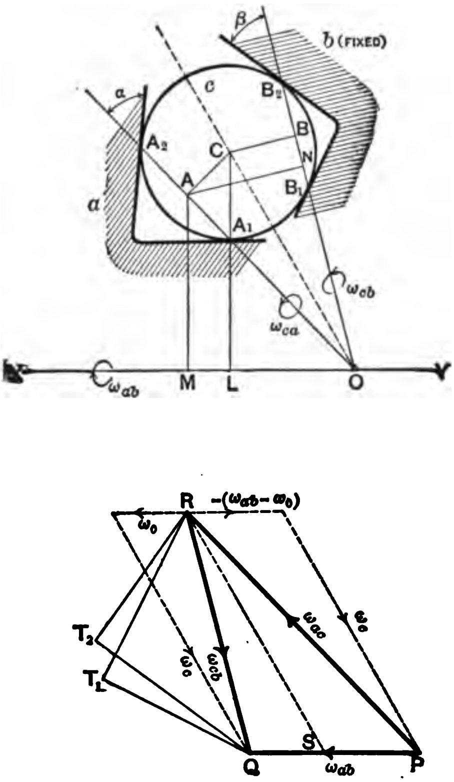

FIG. 229.

and c is rolling upon it about the axis OB, so that we may also say that the linear velocity of the point C is $\omega_{cb} \times CB$, and that of A is $\omega_{cb} \times AN$. Hence,

$$\omega_0 \times CL = \omega_{cb} \times CB,$$
$$\omega_{ab} \times AM = \omega_{cb} \times AN.$$

Thus
$$\frac{\omega_0}{\omega_{ab}} = \frac{CB.AM}{CL.AN}.$$

The angular velocity of the centre of ball relatively to a is readily found if we imagine that the whole bearing has given to it an angular velocity $-\omega_{ab}$. The centre of ball will then be moving with an angular velocity $(\omega_{ab}-\omega_0)$ about the axis XY in the same sense as ω_{c2}, and a will be at rest. Then

$$\omega_{ca}\times CA=(\omega_{ab}-\omega_0)\times CL,$$

$$\omega_{ca}=\frac{CL}{CA}\left(1-\frac{\omega_0}{\omega_{ab}}\right)\omega_{ab},$$

$$\frac{\omega_{ca}}{\omega_{ab}}=\frac{CL}{CA}\cdot\frac{CL.AN-CB.AM}{CL.AN}$$

$$=\frac{CL.AN-CB.AM}{CA.AN}.$$

In the same way we may find an expression for ω_{cb}; for $\omega_{cb}\times AN=\omega_{ab}\times AM$, so that

$$\frac{\omega_{cb}}{\omega_{ab}}=\frac{AM}{AN}.$$

The values of the various angular velocities may also be obtained graphically, as was done in the case of the three-point bearing of Fig. 228. Draw the triangle PQR (Fig. 229) representing ω_{ab}, ω_{bc}, and ω_{ca}. Then we can resolve ω_{cb} into ω_0, the angular velocity of the centre of the ball around XY, and ω_c, the angular velocity about OC. Similarly ω_{ca} can be regarded as the resultant of a velocity ω_c about OC and a velocity $-(\omega_{ab}-\omega_0)$ about XY.

In order to find the spinning and rolling at the points of contact B_1 and B_2, ω_{cb} is to be resolved along and perpendicular to the surfaces of contact, giving

angular velocity of spinning at $B_1=QT_1$;
" " " rolling at $B_1=RT_1$;
" " " spinning at $B_2=RT_2$;
" " " rolling at $B_2=QT_2$.

In the same way the velocities at A_1 and A_2 may be found.

Taking a numerical example, we may determine the various velocities in the three-point bearing shown in Fig. 230, the dimensions being:

Diameter of balls....................	0.25	inch
Distance of ball centre from axis (CL)..	0.5	"
Angle of cone of ball-race b.............	30°	
" α...........................	30°	
" β...........................	20°	

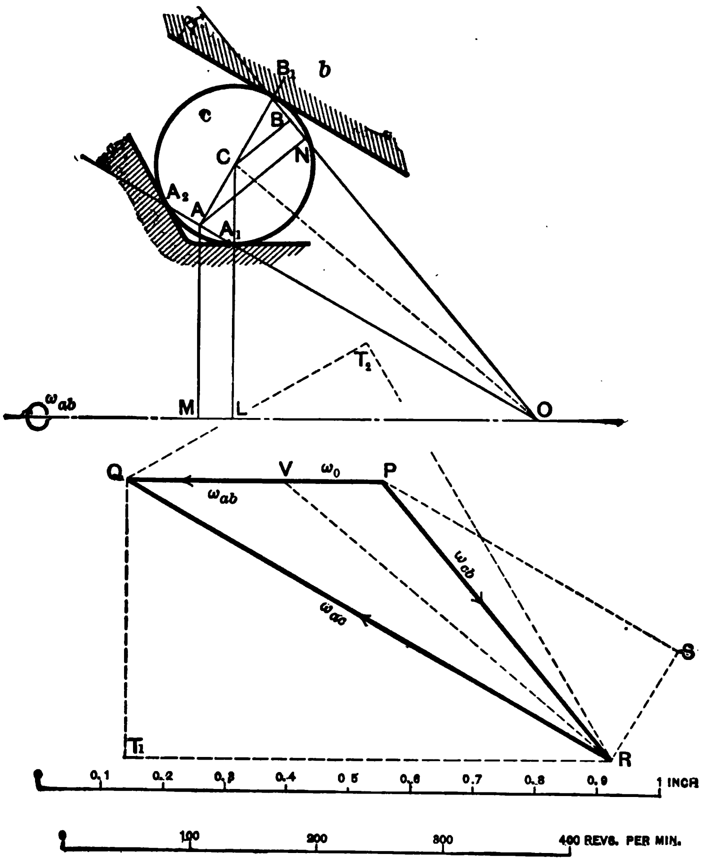

FIG. 230.

After drawing CB, CL, AN, AM, we find $CA=0.108$, $CB=0.115$, $CL=0.400$, $AN=0.218$, $AM=0.309$. Let us suppose that ω_{ab} is 200 revolutions per minute. Then the velocity of centre of balls

$$\omega_0=\omega_{ab}\times\frac{CB.AM}{CL.AN}=\frac{200\times0.115\times0.309}{0.400\times0.218}$$

$$=81.2 \text{ r.p.m. about axis } XY.$$

Again, $$\omega_{ca}=\omega_{ab}\times\left\{\frac{CL}{CA}-\frac{CB.AM}{CA.AN}\right\}$$

$$=200\left\{\frac{0.400}{0.108}-\frac{0.115\times0.309}{0.108\times0.218}\right\}$$

$$=440 \text{ r.p.m. about axis } OA,$$

and

$$\omega_{cb}=200\times\frac{0.309}{0.218}=284 \text{ r.p.m. about } OB.$$

In order to determine the spinning and rolling we have

Angular velocity of spinning at $B_1=\omega_{cb}\sin\beta$
(SR on diagram) $=284\times0.342$
$=100.5$ r.p.m.

Angular velocity of rolling at $B_1=\omega_{cb}\cos\beta$
(SP on diagram) $=284\times0.9396$
$=267$ r.p.m.

Angular velocity of spinning at A_2 or $A_1=\omega_{ac}\sin\alpha$
(QT_2 or QT_1 on diagram) $=440\times0.5$
$=220$ r.p.m.

Angular velocity of rolling at A_2 or $A_1=\omega_{ac}\cos\alpha$
(RT_2 or RT_1 on diagram) $=440\times0.866$
$=381$ r.p.m.

The reader should check these numbers by drawing the velocity diagram for himself, and measuring the various lines representing the velocities.

To compare the relative merits of different ball-bearings we should have to calculate not only the velocities, but also

the pressures between the surfaces at the points of contact; the work wasted in various cases could then be estimated. This part of the work, however, belongs to Dynamics rather than to Kinematics.

For further information on the subject of ball-bearings the reader is referred to Sharp's "Bicycles and Tricycles," Chapter XXV, and to a number of papers in various engineering periodicals.*

* *Engineering*, April 12, 1901. *Zeitschrift d. V. D. I.*, Jan. 27, 1900; *ibid.*, Jan., 1901, pp. 73 and 119; *ibid.*, Jan., 1901, p. 332.

CHAPTER XIII.

KINEMATIC CLASSIFICATION OF MECHANISMS.

101. Historical Sketch. — In treating of the theory of Mechanisms, it has been the aim of many writers to devise some method of analysis whereby mechanical contrivances in general might be resolved into their several component parts, capable of being represented, if necessary, by symbols, and capable also of being recombined in such a fashion as to produce new mechanisms. Such a system, if complete and workable, would of course be of great service to the inventor, and would save him from the fate, only too common, of designing with great toil some device which has been known and used for years. In the words of Willis, "there appears no reason why the construction of a machine for a given purpose should not, like any usual problem, be so reduced to the dominion of the mathematician as to enable him to obtain, by direct and certain methods, all the forms and arrangements that are applicable to the desired purpose, from which he may select at pleasure." It must be confessed that so far no such system of analysis and synthesis has been found of any great practical value; many of the proposals, however, are interesting and suggestive, and a brief account of some of them will not be out of place in this book. Before entering upon it we may glance at the historical development of the subject of the Kinematics of Machines.

A book dating from the eighteenth century* seems to be the first treatise on machines which can be considered at

* Leupold. Theatrum Machinarum. 1724.

all systematic. Leupold's predecessors had indeed described sundry machines and devices, but their order of arrangement was always arbitrary, and no attempt was made to study machines by considering the relative motions of their parts. The theory of machines, treated either from the kinematic or dynamic standpoint, did not in fact exist.

Euler * taught that the motions of rigid bodies should be investigated by the methods of geometry, as well as by the aid of dynamics, but it does not appear that he had in view the special application of these principles to the motions of the parts of machines. Monge in 1794 conceived the idea of treating machines as contrivances for changing one kind of motion into another, and was the first to suggest that the essential "elements of machines" should be enumerated and studied. His system formed the basis of the course adopted in the Ecole Polytechnique soon after its foundation—a course laid out by Lanz and Bétancourt,† and classifying the motions of the parts of machines as (1) rectilinear, (2) circular, or (3) curvilinear. Combinations of these motions are considered. while each motion may be continuous or alternate. The work of Lanz and Bétancourt was incomplete, because no attempt was made to calculate these various motions; their scheme underwent many modifications, and has not survived. A system somewhat similar in intention, but differing in detail, was propounded by Borgnis.‡ It has met with the same fate.

It is to the physicist Ampère § that we owe an important advance. He saw clearly that a mechanism should be studied as "an instrument by the help of which the direction and velocity of a given motion can be altered"; thus going further than Euler, and laying the foundations of that science of Machines to which, in accordance with his suggestion, we apply the name Kinematics.

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* Euler. Theoria Motus Corporum. 1765.

† Lanz and Bétancourt. Essai sur la composition des Machines. 1808.

‡ Borgnis. Traité complet de Mécanique appliquée aux Arts. 1818.

§ Ampère. Essai sur la philosophie des Sciences. 1834.

Ampère was followed by Willis,* who confined himself to the consideration of what he termed the "Elements of Pure Mechanisms," and did not deal with the "generalities of motion." The "Principles of Mechanism" takes a less abstract view of the science of Kinematics than Ampère seems to have held, and in that book the author endeavors to form a system embracing all the elementary combinations of mechanism, and admitting of an investigation of their modifications of motion. He does not attempt to deal with dynamical questions, but gives practical and useful solutions of many leading problems in applied kinematics. His system of classification will receive some consideration in a later section; we shall see that its groundwork is the mode in which the motion is transmitted, or, as we should now express it, the kind of relative motion existing, in various mechanisms.

In several of his books Rankine † deals with kinematical questions, treated under such titles as the Geometry of Machinery and the Theory of Mechanism. His views were in some few respects erroneous and incomplete, and his nomenclature has not been followed to any large extent, but his system of dealing with the motion of machine parts by the aid of instantaneous centres, and his methods of solving certain special problems, were in many cases far more powerful and effective than any previously employed.

The appearance in collected form of the kinematical writings of Reuleaux ‡ furnished students with the first text-book whose methods have met with really wide acceptance. It is to Reuleaux that we owe the idea of a mechanism regarded as a chain made up of links any one of which may be considered as being fixed. Starting with this con-

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* Willis Principles of Mechanism. 1841. (Second Edition 1870.)

† Rankine. Applied Mechanics. 1858
Manual of Machinery and Millwork. 1869

‡ Reuleaux. Theoretische Kinematik. English Translation by Dr. Kennedy. 1876.

ception, and taking account of the relative motion of these links as determined by the pairing of their elements, we are led to a wide and comprehensive view of the whole kinematic theory of mechanisms. The earlier work of Reuleaux has now been supplemented by the publication of a second part of his text-book.*

Burmester's important treatise † is not so well known to English-speaking readers as it should be. Only the first volume, dealing with plane motion, has yet been published. Burmester's method of treatment differs from that of Reuleaux in making a more liberal use of purely mathematical and geometrical principles, but the two authors agree in their fundamental conception of the subject, and, to a large extent, in their nomenclature and definitions. A considerable amount of space is devoted by Burmester to the kinematics of a plane rigid system; he deals with the principles of constraint in plane motion, and passes on to the consideration of plane mechanisms and the relative displacement, velocity, and acceleration of their various parts. The second volume is to treat, after a similar fashion, of non-plane motion.

102. Classification of Willis. Babbage's Notation. — The following sections contain a short account of some of the schemes suggested for classifying and symbolizing the various kinds of mechanisms.

Like almost all his predecessors, Willis contented himself with proposing a scheme of classification without endeavoring to invent any notation, or system of signs, by which a given mechanism could be represented by a formula. Without apparent reason, Willis excludes from his system all hydraulic machines. Some other classes of mechanism, for example those including springs, are also omitted. In fact he considers as "pure mechanisms" only certain types

* **Reuleaux.** Die praktischen Beziehungen der Kinematik zu Geometrie und Mechanik. 1900.

† **Burmester.** Lehrbuch der Kinematik. 1888.

of machines, which seem to have been selected in a somewhat arbitrary fashion. In these machines, according to Willis, motion is transmitted in "elementary combinations" by five methods, namely:

Division.	Method of Transmission.	Example.
A.	By rolling contact.	Toothed gearing of various sorts.
B.	By sliding contact.	Cams, screws, worm- and screw-gearing, escapements.
C.	By wrapping connection.	Bands, chains, and other gearing.
D.	By linkwork.	Cranks, eccentrics, and other linkwork. Ratchet-wheels and clicks.
E.	By reduplication.	Tackle of all sorts.

Each of these five main divisions is again separated into three classes, in which the velocity ratio is either (*a*) constant, (*b*) varying, and (*c*) constant or varying; while due regard is had to the question whether the "directional relation" is constant of varying.

This system or classification has not been widely used, and possesses certain manifest imperfections. It was, however, a great advance on that of Lanz and Bétancourt or on that of Borgnis, because it was designed with a view of facilitating calculations regarding the relative motions, or velocity ratios, in mechanisms, rather than with the aim of classifying mechanisms for purely descriptive purposes.

In the "Principles of Mechanism" Willis devotes some space to the exposition of the scheme of notation proposed by Babbage; * a scheme devised by that ingenious inventor primarily for the purpose of clearly representing the relations of the parts of his calculating-machine, and especially

* A Method of Expressing by Signs the Action of Machinery. Phil. Trans., 1826.

applicable to complex trains of wheel and ratchet gearing. As this notation involves the construction of an elaborate sheet or diagram for each machine, it by no means answers the purpose of a system such as that of Reuleaux, which will be described later, where each mechanism is to be denoted by a formula of three or more symbols. Babbage's method of notation corresponds more closely to that employed by clock and watch-makers, in which the various wheels are represented by the numbers of their teeth, written in successive lines, placing vertically over each other the numbers of wheels which gear together. Thus

$$\begin{array}{lll} 48 & & \\ 6 - & 45 & \\ & 6 - & 30 \end{array}$$

would represent a wheel-train comprising a "great wheel" of 48 teeth gearing with a pinion of 6 teeth, the pinion-arbor or axis carrying a second wheel of 45 teeth, gearing in its turn with a 6-tooth pinion whose arbor carries an escape-wheel of 30 teeth. Babbage, however, shows on his diagram the kind of motion, whether uniform, intermittent, variable, or continuous, of each part with relation to the frame of the machine, and Willis gives an interesting example* of such a diagram, as constructed for a sawmill. It would appear that Babbage's notation, while extremely convenient in certain cases, by no means answers the purpose of a general scheme by means of which the mode of action and relative motions in any given mechanism may be indicated.

103. Classification and Notation of Reuleaux. — Such a system has been devised by Reuleaux,† and is explained and used in his text-book. It is intended to be perfectly general in its application, and includes signs of three kinds, which denote (1) the class or name of the body or link referred

* Principles of Mechanism, Ed. 1870, p. 288.

† Kinematics of Machinery, English Ed., p. 251.

to, as distinguished by its geometrical shape or its nature; (2) the form of the body, whether solid or full, or hollow or open, whether plane or curved; and lastly, (3) the relation of one element to its companion, or of one link to the next in the chain. Some special symbols are required to indicate incomplete pairs, methods of closure, and so on.

In the first division the following *name* symbols have been adopted:

S	Screw.	G	Sphere or globe.
R	Solid of revolution.	A	Sector or arc.
P	Prism.	Z	Tooth or projection.
C	Cylinder.	V	Vessel or chamber.
K	Cone.	T	Tension-organ.
H	Hyperboloid.	Q	Pressure-organ.

These symbols require no explanation.

With regard to the next kind of symbols, those of *form*, it is evidently necessary to indicate among other particulars whether a given body is full, open, or plane; whether its profile is curved or non-circular, or has upon it teeth; or whether its profile or section is prismatic. A link, as we have seen, may be liquid or gaseous, and a large number of other cases may be suggested, all of which should be covered by any general system of symbols. Reuleaux proposes to do this by adding to the Roman capital letters which he selects as the name-symbols, certain form-symbols, written to the right of the name-symbol, and either above or below it. A few examples will illustrate the way of doing this. We may use the following:

$+$	full or solid.	$\circ$	plane.
$-$	open or hollow.	$\sim$	curved profile.

From these and the preceding symbols we have, among many others:

C^+	full cylinder.	C^-	open cylinder.
S^+	screw or bolt.	S^-	nut.

$\tilde{P}^-$ hollow or open prism whose base has a curved outline.
$\tilde{C}_s^+$ non-circular spur-wheel with external teeth.
K_s° face-wheel (plane bevel-wheel).
T_p prismatic tension-organ (flat belt).
T_c circular tension-organ (wire).
Q_λ liquid pressure-organ.
Q_γ gas, air, steam, etc.
V^- cylinder of an engine or pump.
V^+ the piston working in it.

The third class of symbols express relation, as regards pairing, linkage, or fixing, or as regards position and magnitude. Pairing is denoted by a comma, linkage by a dot or dotted line; a fixed link is indicated by underlining, and the usual signs are employed for equality, parallelism, and so on. For example:

$C^- \ldots C^-$ link connecting two open cylinders or eyes.
P_s, C_s^+ rack pairing with a pinion.
$S^+ = S^-$ screw pair, screw and nut being of course equal in size.

Incompleteness is indicated by the use of the sign of division, so that we get:

$\frac{C^-}{2}$ portion of an open cylinder.

$\frac{C^+}{f}$ a full cylinder paired by force-closure.

The method of closure is indicated by the divisor.

As an example of the method of writing out the formula for a simple mechanism, we may refer to Fig. 133. The spur-wheel mechanism *acd* would be written (*d* being the fixed link):

$$\underbrace{C^+ \ldots | \ldots C_s^+}, \; \underbrace{C_s^+ \ldots | \ldots C^+} = \underline{\underbrace{C^- \ldots \| \ldots C^-}}$$

(referring to link *a*, link *b*, and link *d*).

Here | means con-axial; ‖ means parallel.

After describing and enumerating the various symbols, Reuleaux proceeds to show how the resulting formulæ may be shortened. He employs (S), (C), and (P) for a screw pair, a turning pair, and a sliding pair, respectively, and would write $(C_4^{\parallel})\frac{d}{a}$ for "a chain formed of four links, each connecting two parallel cylindric elements"; *d* being the fixed link and *a* the one which drives the chain. This is of course the quadric cylindric crank chain. His symbols have a very wide range of adaptability; the reader will be interested, for example, in the formula for a paddle-steamer, which is

$$C^{+} \ldots | \ldots C_{z}^{+},\ Q_{\lambda} \ldots Q_{\lambda},\ V^{-} \ \ldots + \ldots C_{=}^{-}$$

This may be contracted to $(C^{i}\, C_{z\lambda} V_{\lambda})\frac{b}{a}$.

Here *b* is the liquid link, *a* the paddle-wheels, and *c* the ship itself. V_{λ} is a contraction for V^{-}, Q_{λ}, and $C_{z\lambda}$ for C_{z}, Q_{λ}; + is the sign for "crossed at right angles" when used as the symbol of relation of the elements of a link.

The original text-book must of course be consulted if any real acquaintance with the scheme is desired; the examples given here will serve to indicate the scope and posibilities of the system.

104. Classification of Hearson.—The most recent system of notation devised by Professor Hearson * differs essentially from that of Reuleaux, for it is based on a somewhat different conception of the meaning of the term *machine*. Hearson considers that a machine is to be regarded as "an embodiment of a combination of elementary motions (of which it will be found that the number of kinds is comparatively limited)"; these "elementary motions" being the relative motions of the machine-parts. He treats first of plane mechanisms, and suggests the following symbols:

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* Phil. Trans., 1896, Vol. 187, p. 15.

O a motion of complete and continuous relative rotation.

U a swinging to-and-fro motion, like that of a pendulum, consisting of successive partial rotations in opposite senses.

I a sliding motion.

Taking a four-link mechanism as the general case (three and five or more links being inadmissible in simple machines for reasons already given), it is shown that there may be in such a mechanism either

four O motions,
or three O's and one U } under certain conditions as
or two O's and two U's, } to length of links,
or four U motions;

while it is impossible to have one O motion only and three U's.

On considering the substitution of I motions for O's and U's, it is found that (in all) fourteen combinations of O's, U's, and I's are permissible.

In order to denote such motions as that of the teeth of a pair of spur-wheels, Hearson assigns the symbols

W for a combination of two U motions;
∞ for a combination of two O motions;
C for the wrapping motion of a belt on its pulley.

He further proposes to distinguish between the sense of motions by the use of large and small letters, so that, for example, two pulleys mounted on a frame joined by a crossed belt would be OcCO or oCco; if an open belt were employed the formula would become OcCo or oCcO.

Passing to spherical mechanisms, a similar system is outlined; certain limitations, however, are imposed by the differences between plane and spherical geometrical relations.

Adopting the symbol V for helical motion having a constant pitch ratio, and H for one in which the pitch ratio varies, it is found that we may have the mechanisms UVI, VVI, VVU, and VVV, of which there are eight inversions in

all. With H motions four combinations, with eleven inversions, give us UHI, VHI, VHU, and VHV; these may be classed under the head of cylindrical mechanisms. Hearson proposes to group all remaining simple mechanisms in a fourth division, comprising those in which the axes neither meet nor are parallel. He then discusses compound mechanisms in which there are more than four separately moving pieces, and yet the motions are of a determinate character. This leads to a method of formulating such mechanisms; it will be sufficient here to give as an example the formula for the portion of a locomotive consisting of frame, piston and rod, connecting-rod, crank-axle, coupling-rod, and the crank of a second coupled driving-wheel. This is

$$\mathrm{O} - \mathrm{o} - \mathrm{I} - \mathrm{U} - \mathrm{o} - \mathrm{O}.$$

To explain this formula, it is to be noted that the link shown by the thick line is the frame or the link which is regarded as being fixed. The links which move in contact with it are the piston and rod (U – I), the crank-axle (o – O), and the coupled driving-wheel and its axle (O – o). The connecting-rod will be denoted by (U – o), and the coupling-rod by (O – o), the frame being (o – I – O). Here the large letters refer to turning motions in which the angle is increasing, while small letters indicate those in which the angle is diminishing. The lines connecting the letters of the complete formula show which motions are possessed by the various links. Hearson's scheme does not appear to contemplate the inclusion of fluid links, and, as outlined in the paper cited, is by no means so complete as that of Reuleaux.

105. Remarks on Classification.—It is not surprising that, up to the present, no system of kinematic classification has proved so simple, and at the same time so wide in its scope, as to be generally accepted as an assistance both to the inventor and to the student of the theory of mechanisms. The nomenclature and classification of Reuleaux and the

suggestions of his critics * have rather an academic than a technological interest, and indeed it seems probable that the inevitable complexity of any such scheme, when it finally develops itself, will render it more suitable to the lecture-room than to the drawing-office.

From the instances already quoted it will be seen that kinematicians have taken two distinct points of view in regarding the nature of a machine, and this fundamental divergence has necessarily affected their ideas when searching for a scheme of classification. On the one hand we have the school followed by Hearson and others, who take (with considerable modification) the ideas of Monge and Willis as a basis, and look first at the relative motion of the machine-parts; on the other we have the school of Reuleaux, who consider first the forms of the elements, or working surfaces, which control these relative motions. Whichever line is taken, we are soon driven to the conclusion that in many mechanisms the actual relative motions produced depend not only on the forms of the elements, but also on the way in which the driving forces and the resistances act on the various links of the chain. Reuleaux is thus led to his idea of force-closure of pairs and chains, which has perhaps met with more criticism than any other part of his work, and, to complete his classification, he is obliged to introduce indirectly certain dynamical conditions.

While it is the province of the science of Kinematics of Machines to deal solely with questions of motion, apart from dynamic considerations, it does not seem probable that any really effective system of machine classification can be based simply on kinematic relations.

* For example see Hearson, *loc. cit.*; Koenigs, *Comptes Rendus*, 1901, Aug. 15 and 19, Sept. 2 and 23.

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